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Research Article

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A Framework for Approximation of the Stokes Equations in an Axisymmetric Domain

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Abstract: We develop a framework for solving the stationary, incompressible Stokes equations in an axisymmetric domain. By means of Fourier expansion with respect to the angular variable, the three-dimensional Stokes problem is reduced to an equivalent, countable family of decoupled two-dimensional problems. By using decomposition of three-dimensional Sobolev norms, we derive natural variational spaces for the two-dimensional problems, and show that the variational formulations are well-posed. We analyze the error due to Fourier truncation and conclude that, for data that are sufficiently regular, it suffices to solve a small number of two-dimensional problems.

Keywords: Stokes Equations, Axisymmetric Domain, Weighted Sobolev Space, Fourier Truncation

MSC 2010: 65T99, 76D07

1 Introduction

To determine approximate solutions to fluid flow problems in three-dimensional geometries is a computationally demanding task. In this paper, we present a framework for efficiently solving the stationary, incompressible Stokes equations in an axisymmetric domain $\breve{\Omega}$, which is obtained by rotating its half section Ω around the symmetry axis.

We use Fourier expansions with respect to the angular variable θ, both of the solution and the data, to reduce the three-dimensional Stokes problem to an equivalent, countable family of decoupled two-dimensional problems (set in Ω) for the Fourier coefficients. A natural way to approximate the three-dimensional problem is then to use Fourier truncation and, to obtain a fully discrete scheme, compute approximate solutions to a finite number of the two-dimensional problems.

This is an established technique to approximate boundary value problems that are invariant by rotation. Early error analysis results for second-order elliptic problems can be found in [8] and for Poisson's equation in domains with reentrant edges in [6], both using finite element approximation for the two-dimensional problems. We refer to [5] for additional references to problems described by Laplace or wave equations, the Lamé system, Stokes or Navier–Stokes systems, and Maxwell's equations, and to [6] for early references to algorithms and applications.

We analyze the error due to Fourier truncation and show that, for data that are sufficiently regular with respect to θ, it suffices to solve a small number of two-dimensional problems, which makes the method efficient. Also, the decoupling of the two-dimensional problems makes it suitable for parallel implementation. A further advantage is simplification of the mesh-generation, which is only required for the two-dimensional half section Ω.

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An added complexity is that the natural, variational spaces for the Fourier coefficients turn out to be weighted Sobolev spaces, where the weight is either the distance to the symmetry axis, or its inverse. We derive these spaces by decomposing (through a change of variables to cylindrical coordinates) the three-dimensional norms for the relevant spaces $L^2(\breve{\Omega})$, $L^2_0(\breve{\Omega})$, $(H^1(\breve{\Omega}))^3$, $(H^1_0(\breve{\Omega}))^3$ and $(H^{-1}(\breve{\Omega}))^3$ into sums over all wavenumbers. As a result, we show that the three-dimensional spaces are isometrically isomorphic to a certain subspace of the Cartesian product, over all wavenumbers, of the two-dimensional weighted spaces.

The characterizations of the three-dimensional spaces thus obtained are in agreement with the results in [3], where characterizations of $H^s(\breve{\Omega})$ and $(H^s(\breve{\Omega}))^3$ by Fourier coefficients for any positive real s are derived; here non-integer order spaces are treated first, and the derivation for integer order spaces is then based on Hilbert space interpolation.

As recently shown in [5], the more direct approach for Sobolev spaces of integer order (based on changing to cylindrical coordinates in the three-dimensional norms) results in equivalent norms (compared with the norms in [3]), but where the equivalence constants (unlike in [3]) are independent of the domain. In [5], characterizations of $H^m(\breve{\Omega})$ by Fourier coefficients for any positive integer m are facilitated by the introduction of new differential operators $\partial_\zeta = \frac{1}{\sqrt{2}}(\partial_x - i\partial_y)$ and $\partial_{\bar{\zeta}} = \frac{1}{\sqrt{2}}(\partial_x + i\partial_y)$. The results for vector spaces are then derived from a relation between scalar and vector norms for the Fourier coefficients. In this paper, we work with the differential operators ∂_x and ∂_y and, in the vector case, derive the characterization by directly rewriting the $(H^1(\breve{\Omega}))^3$-norm. We compare our results with [5] in Appendix A. We also refer to the early results in [8], where this direct approach was used to characterize the scalar spaces $H^m(\breve{\Omega})$ for $m = 1, 2, 3$.

The purpose of the present work is to give a comprehensive presentation directly aimed at the Stokes problem providing, inter alia, detailed derivations of the relevant two-dimensional spaces and norms. Taking as starting-point, in fact, Fourier decompositions of the three-dimensional inner products

$$(\cdot\,,\cdot)_{L^2(\breve{\Omega})}, \quad (\cdot\,,\cdot)_{(H^1(\breve{\Omega}))^3} \quad \text{and} \quad (\cdot\,,\cdot)_{(H^1_0(\breve{\Omega}))^3}$$

additionally enables us to derive a decomposition of the negative norm $\|\cdot\|_{(H^{-1}(\breve{\Omega}))^3}$ and to highlight the relation between the three-dimensional weak formulation of the Stokes problem and the two-dimensional weak formulations for the Fourier coefficients.

Examples of how to build on this framework by discretizing the two-dimensional problems can be found in [3], where spectral methods are used, and [2], where two families of finite elements of order 2 (one with continuous pressure corresponding to the Taylor–Hood element and one with discontinuous pressure) are used. The case with an axisymmetric solution (where only the Fourier coefficient of order 0 is considered, and the angular velocity component is equal to zero), has been treated with finite elements in [1, 9, 10].

A paper in preparation will be devoted to design and analysis of stabilized finite elements for the two-dimensional problems.

An outline of the paper is as follows.

- In Section 2, we give some examples of axisymmetric domains and state the stationary, incompressible Stokes equations.
- In Section 3, we recall some basic formulas and state the Stokes problem in cylindrical coordinates.
- In Section 4, we use Fourier expansion with respect to the angular variable to reduce the three-dimensional Stokes problem to a countable family of two-dimensional problems.
- In Section 5, we derive natural variational spaces for the Fourier coefficients by decomposing the relevant three-dimensional norms into sums over all wavenumbers.
- In Section 6, we state variational formulations of the two-dimensional problems and show that these are well-posed.
- In Section 7, we introduce two families of anisotropic spaces that we need to analyze the error due to Fourier truncation.
- In Section 8, we prove an error estimate due to Fourier truncation.

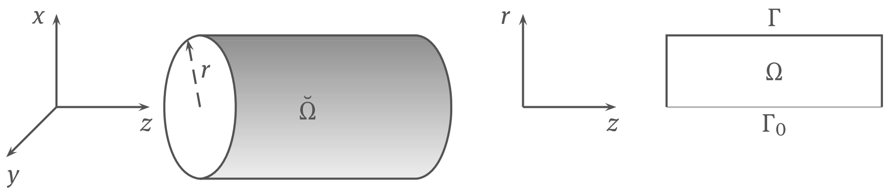

Figure 1: A right circular cylinder. The axisymmetric domain $\breve{\Omega}$ is obtained by rotating its polygonal half section (meridian domain) Ω around the z-axis. The boundary $\partial\Omega = \Gamma \cup \Gamma_0$ of the half section Ω consists of two parts. Γ_0 is the interior of the part of $\partial\Omega$ contained in the z-axis. Rotating the other part, Γ, around the z-axis gives back $\partial\breve{\Omega}$.

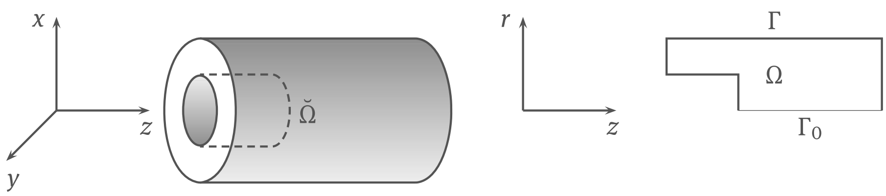

Figure 2: A right circular cylinder with a hole.

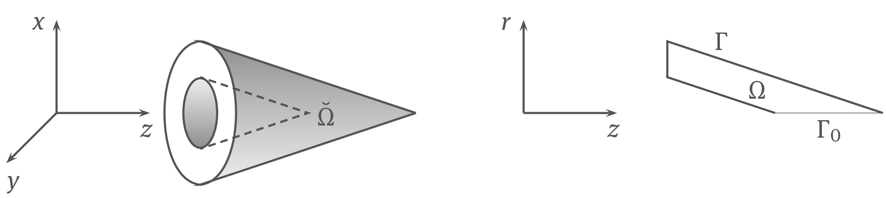

Figure 3: A right circular cone with a hole. Each corner of Ω contained in the z-axis corresponds to a conical singularity in $\partial\breve{\Omega}$, except if the opening angle of Ω at this point is $\pi/2$. Each remaining corner of Ω generates an edge in $\partial\breve{\Omega}$.

2 Model Description

We consider fluid flow in a bounded domain $\breve{\Omega}$ which is invariant by rotation around an axis. We begin by discussing, and give a few examples of, such domains, following the notation in [3]. We then state the stationary, incompressible Stokes equations, which we use to model the flow.

2.1 Axisymmetric Domains

An example of an axisymmetric domain $\breve{\Omega}$ is given in Figure 1. The axisymmetric domain is obtained by rotating its half section (meridian domain) Ω around the symmetry axis. We assume that Ω is polygonal. Note that the boundary $\partial\Omega = \Gamma \cup \Gamma_0$ of the half section Ω consists of two parts where Γ_0, the interior of the part of $\partial\Omega$ contained in the symmetry axis, is a kind of artificial boundary. Two further examples of axisymmetric domains, and their half sections, are given in Figure 2 and Figure 3.

Remark 2.1. In [3], Γ_0 is assumed to be the union of a finite number of segments with positive measure, which means that Ω is not allowed to meet the symmetry axis at isolated points. This assumption, as noted in [5], implies that $\breve{\Omega}$ (as well as its polygonal half section) is a Lipschitz domain, and it guarantees existence of certain trace operators, needed since [3] uses vanishing traces on Γ_0 in the definition of the Fourier coefficient spaces. The more direct approach for integer order Sobolev spaces used in this paper, and more generally in [5], allows for more general axisymmetric domains whose intersection with the symmetry axis is not necessarily a union of intervals and where the trace operators are not well defined.

2.2 Stokes Problem

We model fluid flow through an axisymmetric domain $\breve{\Omega}$ by the stationary, incompressible Stokes equations

$$\begin{cases} -\Delta \underline{\breve{u}} + \underline{\operatorname{grad}}\, \breve{p} = \underline{\breve{f}} & \text{in } \breve{\Omega}, \\ \operatorname{div} \underline{\breve{u}} = 0 & \text{in } \breve{\Omega}, \\ \underline{\breve{u}} = \underline{\breve{g}} & \text{on } \partial\breve{\Omega}, \end{cases} \tag{2.1}$$

where the unknowns are the velocity $\underline{\breve{u}}$ and the pressure $\breve{p}$

$$\underline{\breve{u}} = \breve{u}_x \underline{e}_x + \breve{u}_y \underline{e}_y + \breve{u}_z \underline{e}_z \in (H^1(\breve{\Omega}))^3, \quad \breve{p} \in L_0^2(\breve{\Omega}),$$

and the data are the source term $\underline{\breve{f}}$ and the Dirichlet boundary data $\underline{\breve{g}}$,

$$\underline{\breve{f}} = \breve{f}_x \underline{e}_x + \breve{f}_y \underline{e}_y + \breve{f}_z \underline{e}_z \in (H^{-1}(\breve{\Omega}))^3,$$
$$\underline{\breve{g}} = \breve{g}_x \underline{e}_x + \breve{g}_y \underline{e}_y + \breve{g}_z \underline{e}_z \in (H^{\frac{1}{2}}(\partial\breve{\Omega}))^3.$$

For a vector field $\underline{\breve{v}} = \breve{v}_x \underline{e}_x + \breve{v}_y \underline{e}_y + \breve{v}_z \underline{e}_z$ defined on $\breve{\Omega}$, we will let $\underline{\breve{v}}$ denote both the vector field itself and its Cartesian component vector $(\breve{v}_x, \breve{v}_y, \breve{v}_z)^T$. Note that the divergence-free property $\operatorname{div} \underline{\breve{u}} = 0$ implies a necessary compatibility (zero flux) condition on $\underline{\breve{g}}$:

$$\int_{\partial\breve{\Omega}} \underline{\breve{g}} \cdot \underline{\breve{n}} \, \mathrm{d}\breve{A} = 0, \tag{2.2}$$

where $\underline{\breve{n}}$ denotes the unit outward normal vector to $\breve{\Omega}$ on $\partial\breve{\Omega}$, and $\mathrm{d}\breve{A}$ is (the magnitude of) the area element on $\partial\breve{\Omega}$.

We recall the standard definitions of the Lebesgue and Sobolev spaces (with all derivatives being taken in the sense of distributions)

$$L^2(\breve{\Omega}) := \Big\{ \breve{q} \;\Big|\; \breve{q} : \breve{\Omega} \to \mathbb{C} \text{ measurable}, \int_{\breve{\Omega}} |\breve{q}|^2 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z < +\infty \Big\},$$
$$H^1(\breve{\Omega}) := \{ \breve{v} \mid \breve{v} \in L^2(\breve{\Omega}),\ \partial_x \breve{v} \in L^2(\breve{\Omega}),\ \partial_y \breve{v} \in L^2(\breve{\Omega}),\ \partial_z \breve{v} \in L^2(\breve{\Omega}) \},$$

with norms

$$\|\breve{q}\|_{L^2(\breve{\Omega})} := \Big(\int_{\breve{\Omega}} |\breve{q}|^2 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \Big)^{\frac{1}{2}}, \quad \|\breve{v}\|_{H^1(\breve{\Omega})} := (\|\breve{v}\|^2_{L^2(\breve{\Omega})} + \|\partial_x \breve{v}\|^2_{L^2(\breve{\Omega})} + \|\partial_y \breve{v}\|^2_{L^2(\breve{\Omega})} + \|\partial_z \breve{v}\|^2_{L^2(\breve{\Omega})})^{\frac{1}{2}},$$

and the corresponding inner products $(\cdot\,,\cdot)_{L^2(\breve{\Omega})}$ and $(\cdot\,,\cdot)_{H^1(\breve{\Omega})}$.

Remark 2.2. We consider spaces of complex-valued functions. In Section 4, we will use Fourier expansions of the data and the unknowns to reduce the three-dimensional Stokes problem in $\breve{\Omega}$ to a countable family of two-dimensional problems in the half section Ω. Since the Fourier coefficients (also for real-valued functions) are complex-valued, the two-dimensional variational spaces that we will define in Section 5 will be spaces of complex-valued functions. Since the interplay between the three- and two-dimensional spaces, which will play an important part in what follows, is developed most naturally when all spaces contain complex-valued functions, we will use that also for the three-dimensional spaces, even though in practice the three-dimensional data and solution will be real-valued.

We also recall the subspaces

$$L_0^2(\breve{\Omega}) := \Big\{ \breve{q} \;\Big|\; \breve{q} \in L^2(\breve{\Omega}), \int_{\breve{\Omega}} \breve{q} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = 0 \Big\}, \quad H_0^1(\breve{\Omega}) := \{ \breve{v} \mid \breve{v} \in H^1(\breve{\Omega}),\ \breve{v} = 0 \text{ on } \partial\breve{\Omega} \},$$

and the dual of $H_0^1(\breve{\Omega})$:

$$H^{-1}(\breve{\Omega}) := (H_0^1(\breve{\Omega}))^*.$$

Remark 2.3. By the dual space H^*, we mean the linear space of continuous anti-linear functionals on H so that, in particular, $(u, v)_H = \langle u, v\rangle_{H^*\times H}$ when $u, v \in H$.

On $H_0^1(\breve{\Omega})$, we will use the semi-norm

$$|\breve{v}|_{H^1(\breve{\Omega})} := (\|\partial_x \breve{v}\|^2_{L^2(\breve{\Omega})} + \|\partial_y \breve{v}\|^2_{L^2(\breve{\Omega})} + \|\partial_z \breve{v}\|^2_{L^2(\breve{\Omega})})^{\frac{1}{2}},$$

which is a norm, equivalent to $\|\cdot\|_{H^1(\breve{\Omega})}$, on $H_0^1(\breve{\Omega})$, and the corresponding inner product $(\cdot,\cdot)_{H_0^1(\breve{\Omega})}$.

Denoting by γ_0 the linear and continuous trace operator defined on $H^1(\breve{\Omega})$, we have

$$H^{\frac{1}{2}}(\partial\breve{\Omega}) := \gamma_0(H^1(\breve{\Omega})),$$

with norm

$$\|\breve{g}\|_{H^{\frac{1}{2}}(\partial\breve{\Omega})} := \inf_{\substack{\breve{v}\in H^1(\breve{\Omega})\\ \gamma_0\breve{v}=\breve{g}}} \|\breve{v}\|_{H^1(\breve{\Omega})}.$$

3 Cylindrical Coordinates

Since the domain $\breve{\Omega}$ is axisymmetric, we change from Cartesian coordinates (x, y, z) in $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ to cylindrical coordinates (r, θ, z) in $\mathbb{R}_+ \times (-\pi, \pi] \times \mathbb{R}$, where

$$\begin{cases} r = \sqrt{x^2 + y^2}, \\ \theta = \begin{cases} -\arccos\frac{x}{r} & \text{if } y < 0, \\ \arccos\frac{x}{r} & \text{if } y \geq 0. \end{cases} \end{cases}$$

We define $\tilde{\Omega}$ as the product of the half section Ω and $(-\pi, \pi]$,

$$\tilde{\Omega} := \{(r, \theta, z) \mid (r, z) \in \Omega,\ -\pi < \theta \leq \pi\} \tag{3.1}$$

and, by analogy,

$$\tilde{\Gamma} := \{(r, \theta, z) \mid (r, z) \in \Gamma,\ -\pi < \theta \leq \pi\},$$

where $\tilde{\Omega}$ and $\tilde{\Gamma}$ are point sets in cylindrical coordinates corresponding to the domain $\breve{\Omega}$ and its boundary $\partial\breve{\Omega}$, respectively.

3.1 Basic Formulas

We recall the identities (see Figure 4 for an illustration in the xy-plane)

$$\begin{aligned} \partial_x &= \cos\theta\partial_r - \frac{1}{r}\sin\theta\partial_\theta, & \partial_y &= \sin\theta\partial_r + \frac{1}{r}\cos\theta\partial_\theta, \\ \underline{e}_x &= \cos\theta\,\underline{e}_r - \sin\theta\,\underline{e}_\theta, & \underline{e}_y &= \sin\theta\,\underline{e}_r + \cos\theta\,\underline{e}_\theta, \end{aligned} \tag{3.2}$$

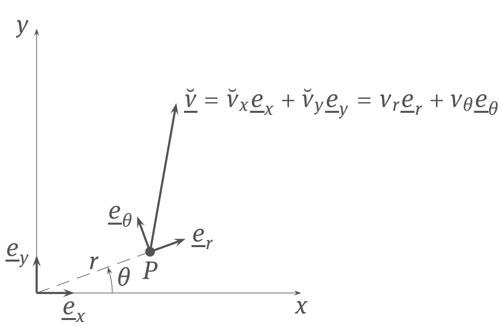

Figure 4: A point P with Cartesian coordinates (x, y) and cylindrical coordinates (r, θ), where $x = r\cos\theta$ and $y = r\sin\theta$. A vector $\underline{\breve{v}}$, with tail in P, with Cartesian components $(\breve{v}_x, \breve{v}_y)^T$ and cylindrical components $(v_r, v_\theta)^T$.

relating the partial derivatives and the orthonormal basis vectors of the Cartesian and cylindrical coordinate systems, and

$$\check{v}_x = \cos\theta v_r - \sin\theta v_\theta, \quad \check{v}_y = \sin\theta v_r + \cos\theta v_\theta, \quad \check{v}_z = v_z, \tag{3.3}$$

relating the component vectors $\underline{\check{v}} = (\check{v}_x, \check{v}_y, \check{v}_z)^T$ and $\underline{v} = (v_r, v_\theta, v_z)^T$ of a vector field

$$\underline{\check{v}} = \check{v}_x \underline{e}_x + \check{v}_y \underline{e}_y + \check{v}_z \underline{e}_z = v_r \underline{e}_r + v_\theta \underline{e}_\theta + v_z \underline{e}_z$$

expressed in the two coordinate systems. We will write (3.3) in matrix form $\underline{\check{v}} = \mathcal{R}_\theta \underline{v}$, where

$$\mathcal{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

From these identities follow the formulas for the gradient and the Laplacian operator acting on a scalar function $\check{v}(x, y, z) = v(r, \theta, z)$,

$$\underline{\operatorname{grad}}\, \check{v} = \partial_x \check{v} \underline{e}_x + \partial_y \check{v} \underline{e}_y + \partial_z \check{v} \underline{e}_z = \partial_r v \underline{e}_r + \frac{1}{r} \partial_\theta v \underline{e}_\theta + \partial_z v \underline{e}_z, \tag{3.4}$$

$$\Delta \check{v} = \partial_x^2 \check{v} + \partial_y^2 \check{v} + \partial_z^2 \check{v} = \partial_r^2 v + \frac{1}{r} \partial_r v + \frac{1}{r^2} \partial_\theta^2 v + \partial_z^2 v = \frac{1}{r} \partial_r (r \partial_r v) + \frac{1}{r^2} \partial_\theta^2 v + \partial_z^2 v \tag{3.5}$$

and, for the divergence and the vector Laplacian operator acting on a vector function $\underline{\check{v}}(x, y, z) = \mathcal{R}_\theta \underline{v}(r, \theta, z)$,

$$\operatorname{div} \underline{\check{v}} = \partial_x \check{v}_x + \partial_y \check{v}_y + \partial_z \check{v}_z = \partial_r v_r + \frac{1}{r} v_r + \frac{1}{r} \partial_\theta v_\theta + \partial_z v_z = \frac{1}{r} \partial_r (r v_r) + \frac{1}{r} \partial_\theta v_\theta + \partial_z v_z, \tag{3.6}$$

$$\begin{aligned} \Delta \underline{\check{v}} &= (\partial_x^2 \check{v}_x + \partial_y^2 \check{v}_x + \partial_z^2 \check{v}_x) \underline{e}_x + (\partial_x^2 \check{v}_y + \partial_y^2 \check{v}_y + \partial_z^2 \check{v}_y) \underline{e}_y + (\partial_x^2 \check{v}_z + \partial_y^2 \check{v}_z + \partial_z^2 \check{v}_z) \underline{e}_z \\ &= \Big(\frac{1}{r} \partial_r (r \partial_r v_r) + \frac{1}{r^2} \partial_\theta^2 v_r + \partial_z^2 v_r - \frac{1}{r^2} v_r - \frac{2}{r^2} \partial_\theta v_\theta \Big) \underline{e}_r \\ &\quad + \Big(\frac{1}{r} \partial_r (r \partial_r v_\theta) + \frac{1}{r^2} \partial_\theta^2 v_\theta + \partial_z^2 v_\theta - \frac{1}{r^2} v_\theta + \frac{2}{r^2} \partial_\theta v_r \Big) \underline{e}_\theta + \Big(\frac{1}{r} \partial_r (r \partial_r v_z) + \frac{1}{r^2} \partial_\theta^2 v_z + \partial_z^2 v_z \Big) \underline{e}_z, \end{aligned} \tag{3.7}$$

where the last two terms in the radial and angular components of $\Delta \underline{\check{v}}$ result from the θ-dependence of $\underline{e}_r = \cos\theta\, \underline{e}_x + \sin\theta\, \underline{e}_y$ and $\underline{e}_\theta = -\sin\theta\, \underline{e}_x + \cos\theta\, \underline{e}_y$, by the relations $\partial_\theta \underline{e}_r = \underline{e}_\theta$ and $\partial_\theta \underline{e}_\theta = -\underline{e}_r$.

3.2 Stokes Problem in Cylindrical Coordinates

Expressing both the data and the unknowns in cylindrical coordinates

$$\begin{aligned} \underline{\check{u}} &= \check{u}_x \underline{e}_x + \check{u}_y \underline{e}_y + \check{u}_z \underline{e}_z = u_r \underline{e}_r + u_\theta \underline{e}_\theta + u_z \underline{e}_z, \\ \check{p} &= p, \\ \underline{\check{f}} &= \check{f}_x \underline{e}_x + \check{f}_y \underline{e}_y + \check{f}_z \underline{e}_z = f_r \underline{e}_r + f_\theta \underline{e}_\theta + f_z \underline{e}_z, \\ \underline{\check{g}} &= \check{g}_x \underline{e}_x + \check{g}_y \underline{e}_y + \check{g}_z \underline{e}_z = g_r \underline{e}_r + g_\theta \underline{e}_\theta + g_z \underline{e}_z, \end{aligned}$$

where $\underline{u} = (u_r, u_\theta, u_z)^T$, p, and $\underline{f} = (f_r, f_\theta, f_z)^T$ are functions (distributions) on $\tilde{\Omega}$ and $\underline{g} = (g_r, g_\theta, g_z)^T$ on $\tilde{\Gamma}$, from (3.4)–(3.7), we can write the Stokes problem (2.1) in cylindrical coordinates

$$\begin{cases} -\Delta u_r + \dfrac{1}{r^2} u_r + \dfrac{2}{r^2} \partial_\theta u_\theta + \partial_r p = f_r & \text{in } \tilde{\Omega}, \\ -\Delta u_\theta + \dfrac{1}{r^2} u_\theta - \dfrac{2}{r^2} \partial_\theta u_r + \dfrac{1}{r} \partial_\theta p = f_\theta & \text{in } \tilde{\Omega}, \\ -\Delta u_z + \partial_z p = f_z & \text{in } \tilde{\Omega}, \\ \dfrac{1}{r} \partial_r (r u_r) + \dfrac{1}{r} \partial_\theta u_\theta + \partial_z u_z = 0 & \text{in } \tilde{\Omega}, \\ \underline{u} = \underline{g} & \text{on } \tilde{\Gamma}, \end{cases} \tag{3.8}$$

where, from (3.5),

$$\Delta v = \frac{1}{r} \partial_r (r \partial_r v) + \frac{1}{r^2} \partial_\theta^2 v + \partial_z^2 v.$$

4 Fourier Expansion

A natural way to reduce the three-dimensional Stokes problem (3.8) in $\tilde{\Omega}$ to a countable family of two-dimensional problems in Ω is to use Fourier expansion with respect to the angular variable θ, both of the solution

$$\underline{u}(r,\theta,z) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \underline{u}^k(r,z)e^{ik\theta}, \tag{4.1}$$

$$p(r,\theta,z) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} p^k(r,z)e^{ik\theta}, \tag{4.2}$$

where $\underline{u}^k = (u_r^k, u_\theta^k, u_z^k)^T$, and of the data

$$\underline{f}(r,\theta,z) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \underline{f}^k(r,z)e^{ik\theta}, \tag{4.3}$$

$$\underline{g}(r,\theta,z) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \underline{g}^k(r,z)e^{ik\theta}. \tag{4.4}$$

4.1 Two-Dimensional Problems

Inserting the Fourier expansions (4.1)–(4.4) into (3.8) results, since the Stokes problem is linear and invariant by rotation (which means that the coefficients of the Stokes operator in cylindrical coordinates do not depend on θ), in uncoupled two-dimensional problems for each Fourier coefficient pair $(\underline{u}^k, p^k)$, $k \in \mathbb{Z}$:

$$\begin{cases} -\Delta_a u_r^k + \dfrac{1+k^2}{r^2}u_r^k + \dfrac{2ik}{r^2}u_\theta^k + \partial_r p^k = f_r^k & \text{in } \Omega, \\ -\Delta_a u_\theta^k + \dfrac{1+k^2}{r^2}u_\theta^k - \dfrac{2ik}{r^2}u_r^k + \dfrac{ik}{r}p^k = f_\theta^k & \text{in } \Omega, \\ -\Delta_a u_z^k + \dfrac{k^2}{r^2}u_z^k + \partial_z p^k = f_z^k & \text{in } \Omega, \\ \operatorname{div}_k \underline{u}^k = 0 & \text{in } \Omega, \\ \underline{u}^k = \underline{g}^k & \text{on } \Gamma, \end{cases} \tag{4.5}$$

where Δ_a denotes the axisymmetric part of Δ:

$$\Delta_a v := \frac{1}{r}\partial_r(r\partial_r v) + \partial_z^2 v,$$

and

$$\operatorname{div}_k \underline{u}^k := \frac{1}{r}\partial_r(ru_r^k) + \frac{ik}{r}u_\theta^k + \partial_z u_z^k. \tag{4.6}$$

Remark 4.1. We will, for all $k \in \mathbb{Z}$, show existence and uniqueness of solutions to (4.5) in Section 6.2.

Remark 4.2. By taking the complex conjugate of (4.5), it is easy to see that, for real-valued data $\underline{f}$ and $\underline{g}$, in which case (letting $\bar{\underline{f}}^k$, with some ambiguity of notation, denote the complex conjugate of $\underline{f}^k$)

$$\underline{f}^{-k} = \bar{\underline{f}}^k, \quad \underline{g}^{-k} = \bar{\underline{g}}^k,$$

the pair $(\bar{\underline{u}}^k, \bar{p}^k)$ solves (4.5) with k replaced by $-k$. This means that the Fourier coefficients of the solution will also satisfy

$$\underline{u}^{-k} = \bar{\underline{u}}^k, \quad p^{-k} = \bar{p}^k$$

(corresponding, of course, to a unique, real-valued solution of the three-dimensional Stokes problem for real-valued data), so in the practical case with real-valued data, we only need to solve the problems (4.5) for $k \geq 0$ and, in addition, the solution for $k = 0$ will be real-valued.

Remark 4.3. The compatibility condition (2.2) translates into a condition on the Fourier coefficient $\underline{g}^0$:

$$\int_\Gamma (g_r^0 n_r + g_z^0 n_z) r \,\mathrm{d}s = 0, \tag{4.7}$$

where $\underline{n} = (n_r, n_z)^T$ denotes the unit outward normal vector to Ω on Γ, and $\mathrm{d}s$ is (the length of) the line element along Γ.

5 Variational Spaces

In Section 6.1, we will state variational formulations of the two-dimensional problems (4.5). To determine natural variational spaces for the Fourier coefficients defined on the half section Ω, we start by expressing the $L^2(\breve{\Omega})$- and $(H^1(\breve{\Omega}))^3$-inner products in cylindrical coordinates, as integrals over $\tilde{\Omega}$, and use Fourier expansions to derive decompositions of the inner products, and associated norms, into sums over all wavenumbers.

Based on the structure of the different terms, which are weighted integrals over Ω of the Fourier coefficients and their derivatives, we define weighted Sobolev spaces on Ω.

As a result, we obtain characterizations of the three-dimensional spaces $L^2(\breve{\Omega})$ and $(H^1(\breve{\Omega}))^3$, in terms of two-dimensional weighted spaces on Ω for all Fourier coefficients. In particular, we show that the three-dimensional spaces are isometrically isomorphic to a subspace of the Cartesian product, over all wavenumbers, of the two-dimensional weighted spaces. As a corollary, we obtain corresponding characterizations of the subspaces $L^2_0(\breve{\Omega})$ and $(H^1_0(\breve{\Omega}))^3$.

Using the results for $(H^1_0(\breve{\Omega}))^3$, we finally derive a characterization (also in terms of spaces for all Fourier coefficients) of its dual $(H^{-1}(\breve{\Omega}))^3$.

5.1 Fourier Decomposition of Inner Products and Norms

In cylindrical coordinates, the $L^2(\breve{\Omega})$-inner product of two scalar functions

$$\breve{p}(x, y, z) = p(r, \theta, z), \quad \breve{q}(x, y, z) = q(r, \theta, z) \tag{5.1}$$

is expressed as an integral over $\tilde{\Omega}$, defined by (3.1), with weight r:

$$(\breve{p}, \breve{q})_{L^2(\breve{\Omega})} = \int_{\breve{\Omega}} \breve{p}\bar{\breve{q}} \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z = \int_{\tilde{\Omega}} p\bar{q} r \,\mathrm{d}r \,\mathrm{d}\theta \,\mathrm{d}z. \tag{5.2}$$

For two vector functions

$$\underline{\breve{u}}(x, y, z) = \mathcal{R}_\theta \underline{u}(r, \theta, z), \quad \underline{\breve{v}}(x, y, z) = \mathcal{R}_\theta \underline{v}(r, \theta, z), \tag{5.3}$$

the $(H^1(\breve{\Omega}))^3$-inner product

$$\begin{aligned}
(\underline{\breve{u}}, \underline{\breve{v}})_{(H^1(\breve{\Omega}))^3} &= \int_{\breve{\Omega}} (\underline{\breve{u}} \cdot \bar{\underline{\breve{v}}} + \underline{\underline{\operatorname{grad}}}\, \underline{\breve{u}} : \underline{\underline{\operatorname{grad}}}\, \bar{\underline{\breve{v}}}) \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z \\
&= \int_{\breve{\Omega}} \big(\breve{u}_x \bar{\breve{v}}_x + \breve{u}_y \bar{\breve{v}}_y + \breve{u}_z \bar{\breve{v}}_z \\
&\qquad + \partial_x \breve{u}_x \partial_x \bar{\breve{v}}_x + \partial_y \breve{u}_x \partial_y \bar{\breve{v}}_x + \partial_z \breve{u}_x \partial_z \bar{\breve{v}}_x \\
&\qquad + \partial_x \breve{u}_y \partial_x \bar{\breve{v}}_y + \partial_y \breve{u}_y \partial_y \bar{\breve{v}}_y + \partial_z \breve{u}_y \partial_z \bar{\breve{v}}_y \\
&\qquad + \partial_x \breve{u}_z \partial_x \bar{\breve{v}}_z + \partial_y \breve{u}_z \partial_y \bar{\breve{v}}_z + \partial_z \breve{u}_z \partial_z \bar{\breve{v}}_z\big) \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z
\end{aligned}$$

can, through repeated use of relations (3.2), (3.3) and the Pythagorean trigonometric identity, be expressed in cylindrical coordinates:

$$
\begin{aligned}
(\underline{\breve{u}}, \underline{\breve{v}})_{(H^1(\breve{\Omega}))^3} = \int\limits_{\breve{\Omega}} \Big(& u_r \bar{v}_r + u_\theta \bar{v}_\theta + u_z \bar{v}_z \\
& + \partial_r u_r \partial_r \bar{v}_r + \frac{1}{r^2} \partial_\theta u_r \partial_\theta \bar{v}_r + \partial_z u_r \partial_z \bar{v}_r + \frac{1}{r^2} u_r \bar{v}_r \\
& + \partial_r u_\theta \partial_r \bar{v}_\theta + \frac{1}{r^2} \partial_\theta u_\theta \partial_\theta \bar{v}_\theta + \partial_z u_\theta \partial_z \bar{v}_\theta + \frac{1}{r^2} u_\theta \bar{v}_\theta \\
& + \frac{1}{r^2} \big((\partial_\theta u_\theta) \bar{v}_r + u_r (\partial_\theta \bar{v}_\theta) - (\partial_\theta u_r) \bar{v}_\theta - u_\theta (\partial_\theta \bar{v}_r) \big) \\
& + \partial_r u_z \partial_r \bar{v}_z + \frac{1}{r^2} \partial_\theta u_z \partial_\theta \bar{v}_z + \partial_z u_z \partial_z \bar{v}_z \Big) r \,\mathrm{d}r \,\mathrm{d}\theta \,\mathrm{d}z.
\end{aligned}
\tag{5.4}
$$

We now consider Fourier expansions

$$
p = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} p^k(r, z) e^{ik\theta}, \quad q = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} q^k(r, z) e^{ik\theta}, \tag{5.5}
$$

$$
\underline{u} = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} \underline{u}^k(r, z) e^{ik\theta}, \quad \underline{v} = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} \underline{v}^k(r, z) e^{ik\theta}, \tag{5.6}
$$

where $\underline{v} = (v_r, v_\theta, v_z)^T$ and $\underline{v}^k = (v_r^k, v_\theta^k, v_z^k)^T$. Using the orthogonality on $[-\pi, \pi]$ of the family $\{e^{ik\theta}\}_{k=-\infty}^{+\infty}$ of basis functions, we obtain (letting $\bar{q}^k$, as before, denote the complex conjugate of q^k)

$$
\begin{aligned}
\int\limits_{-\pi}^{\pi} p \bar{q} \,\mathrm{d}\theta &= \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \Big(\sum_{k \in \mathbb{Z}} p^k(r, z) e^{ik\theta} \Big) \Big(\sum_{k' \in \mathbb{Z}} \bar{q}^{k'}(r, z) e^{-ik'\theta} \Big) \,\mathrm{d}\theta \\
&= \frac{1}{2\pi} \sum_{k, k' \in \mathbb{Z}} p^k(r, z) \bar{q}^{k'}(r, z) \int\limits_{-\pi}^{\pi} e^{i(k-k')\theta} \,\mathrm{d}\theta = \sum_{k \in \mathbb{Z}} p^k(r, z) \bar{q}^k(r, z)
\end{aligned}
\tag{5.7}
$$

and, similarly, in four other representative cases,

$$
\begin{aligned}
&\int\limits_{-\pi}^{\pi} \partial_r u_r \partial_r \bar{v}_r \,\mathrm{d}\theta = \sum_{k \in \mathbb{Z}} \partial_r u_r^k(r, z) \partial_r \bar{v}_r^k(r, z), \quad && \int\limits_{-\pi}^{\pi} \partial_\theta u_r \partial_\theta \bar{v}_r \,\mathrm{d}\theta = \sum_{k \in \mathbb{Z}} k^2 u_r^k(r, z) \bar{v}_r^k(r, z), \\
&\int\limits_{-\pi}^{\pi} (\partial_\theta u_\theta) \bar{v}_r \,\mathrm{d}\theta = \sum_{k \in \mathbb{Z}} ik u_\theta^k(r, z) \bar{v}_r^k(r, z), && \int\limits_{-\pi}^{\pi} u_\theta (\partial_\theta \bar{v}_r) \,\mathrm{d}\theta = \sum_{k \in \mathbb{Z}} u_\theta^k(r, z)(-ik) \bar{v}_r^k(r, z).
\end{aligned}
\tag{5.8}
$$

Inserting (5.7) and (5.8) (and analogous results for the remaining θ-integrals) into (5.2), (5.4) gives

$$
\begin{aligned}
(\breve{p}, \breve{q})_{L^2(\breve{\Omega})} &= \sum_{k \in \mathbb{Z}} \int\limits_{\Omega} p^k \bar{q}^k r \,\mathrm{d}r \,\mathrm{d}z, \\
(\underline{\breve{u}}, \underline{\breve{v}})_{(H^1(\breve{\Omega}))^3} &= \sum_{k \in \mathbb{Z}} \int\limits_{\Omega} \Big(u_r^k \bar{v}_r^k + u_\theta^k \bar{v}_\theta^k + u_z^k \bar{v}_z^k \\
&\qquad + \partial_r u_r^k \partial_r \bar{v}_r^k + \partial_z u_r^k \partial_z \bar{v}_r^k + \frac{1 + k^2}{r^2} u_r^k \bar{v}_r^k + \frac{2ik}{r^2} u_\theta^k \bar{v}_r^k \\
&\qquad + \partial_r u_\theta^k \partial_r \bar{v}_\theta^k + \partial_z u_\theta^k \partial_z \bar{v}_\theta^k + \frac{1 + k^2}{r^2} u_\theta^k \bar{v}_\theta^k - \frac{2ik}{r^2} u_r^k \bar{v}_\theta^k \\
&\qquad + \partial_r u_z^k \partial_r \bar{v}_z^k + \partial_z u_z^k \partial_z \bar{v}_z^k + \frac{k^2}{r^2} u_z^k \bar{v}_z^k \Big) r \,\mathrm{d}r \,\mathrm{d}z,
\end{aligned}
\tag{5.9}
$$

expressing the $L^2(\breve{\Omega})$-inner product and the $(H^1(\breve{\Omega}))^3$-inner product as sums, over all wavenumbers, of weighted integrals over the half section Ω of the Fourier coefficients and their derivatives.

The corresponding decompositions of the associated norms are

$$\|\breve{q}\|^2_{L^2(\breve{\Omega})} = \sum_{k\in\mathbb{Z}} \int_\Omega |q^k|^2 r \,\mathrm{d}r\,\mathrm{d}z, \tag{5.10}$$

$$\begin{aligned}\|\breve{\underline{v}}\|^2_{(H^1(\breve{\Omega}))^3} = \sum_{k\in\mathbb{Z}} \int_\Omega \Big(& |v_r^k|^2 + |v_\theta^k|^2 + |v_z^k|^2 \\ & + |\partial_r v_r^k|^2 + |\partial_z v_r^k|^2 + \frac{1+k^2}{r^2}|v_r^k|^2 + \frac{2ik}{r^2} v_\theta^k \bar{v}_r^k \\ & + |\partial_r v_\theta^k|^2 + |\partial_z v_\theta^k|^2 + \frac{1+k^2}{r^2}|v_\theta^k|^2 - \frac{2ik}{r^2} v_r^k \bar{v}_\theta^k \\ & + |\partial_r v_z^k|^2 + |\partial_z v_z^k|^2 + \frac{k^2}{r^2}|v_z^k|^2 \Big) r\,\mathrm{d}r\,\mathrm{d}z.\end{aligned} \tag{5.11}$$

5.2 Weighted Sobolev Spaces on Ω

Led by (5.10) and (5.11), where each term is an integral over Ω with weight r or r^{-1}, we first introduce the spaces

$$L^2_1(\Omega) := \Big\{ v \;\Big|\; v\colon \Omega \to \mathbb{C} \text{ measurable}, \int_\Omega |v(r,z)|^2 r\,\mathrm{d}r\,\mathrm{d}z < +\infty \Big\},$$

$$L^2_{-1}(\Omega) := \Big\{ v \;\Big|\; v\colon \Omega \to \mathbb{C} \text{ measurable}, \int_\Omega |v(r,z)|^2 r^{-1}\,\mathrm{d}r\,\mathrm{d}z < +\infty \Big\},$$

equipped with the natural norms

$$\|v\|_{L^2_1(\Omega)} := \Big(\int_\Omega |v(r,z)|^2 r\,\mathrm{d}r\,\mathrm{d}z\Big)^{\frac12}, \quad \|v\|_{L^2_{-1}(\Omega)} := \Big(\int_\Omega |v(r,z)|^2 r^{-1}\,\mathrm{d}r\,\mathrm{d}z\Big)^{\frac12}.$$

Next, we define $H^1_1(\Omega)$ as the space of functions in $L^2_1(\Omega)$ such that their partial derivatives (being taken in the sense of distributions) of order 1 belong to $L^2_1(\Omega)$, equipped with the semi-norm

$$|v|_{H^1_1(\Omega)} := (\|\partial_r v\|^2_{L^2_1(\Omega)} + \|\partial_z v\|^2_{L^2_1(\Omega)})^{\frac12}$$

and norm

$$\|v\|_{H^1_1(\Omega)} := (\|v\|^2_{L^2_1(\Omega)} + |v|^2_{H^1_1(\Omega)})^{\frac12}.$$

Remark 5.1. The definition can be extended in a natural way to $H^m_1(\Omega)$ for an arbitrary integer $m \geq 2$ and further, by interpolation, to $H^s_1(\Omega)$ for non-integer $s > 0$.

We will also need the weighted space

$$V^1_1(\Omega) := H^1_1(\Omega) \cap L^2_{-1}(\Omega),$$

equipped with the norm

$$\|v\|_{V^1_1(\Omega)} := (\|v\|^2_{L^2_{-1}(\Omega)} + |v|^2_{H^1_1(\Omega)})^{\frac12}.$$

It can be proved [8, Proposition 4.1] that all functions in $V^1_1(\Omega)$ have a null trace on the part Γ_0 of the boundary contained in the z-axis.

We finally introduce the subspaces

$$L^2_{1,0}(\Omega) := \Big\{ q \;\Big|\; q \in L^2_1(\Omega), \int_\Omega q(r,z) r\,\mathrm{d}r\,\mathrm{d}z = 0 \Big\}, \tag{5.12}$$

consisting of functions in $L^2_1(\Omega)$ with weighted integral equal to zero, and

$$H^1_{1\diamond}(\Omega) := \{ v \mid v \in H^1_1(\Omega),\ v = 0 \text{ on } \Gamma \}, \quad V^1_{1\diamond}(\Omega) := \{ v \mid v \in V^1_1(\Omega),\ v = 0 \text{ on } \Gamma \},$$

consisting of functions in $H^1_1(\Omega)$ and $V^1_1(\Omega)$ that vanish on the part $\Gamma = \partial\Omega \backslash \Gamma_0$ of the boundary that is not contained in the z-axis.

All spaces defined above are Hilbert spaces for the inner products associated with the given norms.

5.3 Characterization of $L^2(\breve{\Omega})$ and $L_0^2(\breve{\Omega})$

Recalling identity (5.10), we let the right-hand side terms, for all $k \in \mathbb{Z}$, define spaces $L^2_{(k)}(\Omega) := L^2_1(\Omega)$ for the Fourier coefficients q^k of a function $\breve{q} \in L^2(\breve{\Omega})$, with norms

$$\|q^k\|^2_{L^2_{(k)}(\Omega)} = \|q^k\|^2_{L^2_1(\Omega)} = \int_\Omega |q^k|^2 r \, \mathrm{d}r \, \mathrm{d}z. \tag{5.13}$$

Introducing the l_2-sum (see [7, p. 63]) $\bigoplus_2\{L^2_{(k)}(\Omega) \mid k \in \mathbb{Z}\}$, which is the subspace of the Cartesian product $\times\{L^2_{(k)}(\Omega) \mid k \in \mathbb{Z}\}$ consisting of all sequences $(q^k)_{k\in\mathbb{Z}}$ for which the norm

$$\|(q^k)\|_{\bigoplus_2\{L^2_{(k)}(\Omega)|k\in\mathbb{Z}\}} := \Big(\sum_{k\in\mathbb{Z}} \|q^k\|^2_{L^2_{(k)}(\Omega)}\Big)^{\frac{1}{2}} < +\infty, \tag{5.14}$$

and combining (5.10) with (5.13)–(5.14), we obtain the following characterization of $L^2(\breve{\Omega})$.

Theorem 5.2. *The mapping $\breve{q} \mapsto (q^k)_{k\in\mathbb{Z}}$, defined by (5.1) and (5.5), is an isometric isomorphism between $L^2(\breve{\Omega})$ and the l_2-sum $\bigoplus_2\{L^2_{(k)}(\Omega) \mid k \in \mathbb{Z}\}$:*

$$\|\breve{q}\|_{L^2(\breve{\Omega})} = \Big(\sum_{k\in\mathbb{Z}} \|q^k\|^2_{L^2_{(k)}(\Omega)}\Big)^{\frac{1}{2}} = \|(q^k)\|_{\bigoplus_2\{L^2_{(k)}(\Omega)|k\in\mathbb{Z}\}}, \tag{5.15}$$

where, for all $k \in \mathbb{Z}$, $L^2_{(k)}(\Omega) = L^2_1(\Omega)$.

Since, for $\breve{q} \in L^2(\breve{\Omega})$, by (5.1) and (5.5),

$$\int_{\breve{\Omega}} \breve{q}(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_{\tilde{\Omega}} q(r,\theta,z) r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \int_{\tilde{\Omega}} q^k(r,z)e^{ik\theta} r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z = \sqrt{2\pi}\int_\Omega q^0(r,z) r\,\mathrm{d}r\,\mathrm{d}z,$$

we also, recalling (5.12), obtain a characterization of the subspace $L^2_0(\breve{\Omega})$ of $L^2(\breve{\Omega})$.

Corollary 5.3. *The mapping $\breve{q} \mapsto (q^k)_{k\in\mathbb{Z}}$, defined by (5.1) and (5.5), is an isometric isomorphism between $L^2_0(\breve{\Omega})$ and the l_2-sum $\bigoplus_2\{L^2_{(k),0}(\Omega) \mid k \in \mathbb{Z}\}$, where*

$$L^2_{(k),0}(\Omega) := \begin{cases} L^2_{1,0}(\Omega) & \text{if } k = 0, \\ L^2_1(\Omega) & \text{if } k \neq 0, \end{cases}$$

and, for all $k \in \mathbb{Z}$, $\|q^k\|_{L^2_{(k),0}(\Omega)} = \|q^k\|_{L^2_1(\Omega)}$.

5.4 Characterization of $(H^1(\breve{\Omega}))^3$ and $(H^1_0(\breve{\Omega}))^3$

Similarly, led by the right-hand side terms in identity (5.11), for all $k \in \mathbb{Z}$, we define spaces $\boldsymbol{H}^1_{(k)}(\Omega)$ for the Fourier coefficient triples $\underline{v}^k = (v^k_r, v^k_\theta, v^k_z)^T$ of a vector function $\underline{\breve{v}} \in (H^1(\breve{\Omega}))^3$, by the norms

$$\begin{aligned} \|\underline{v}^k\|^2_{\boldsymbol{H}^1_{(k)}(\Omega)} := \int_\Omega \Big(& |v^k_r|^2 + |v^k_\theta|^2 + |v^k_z|^2 \\ & + |\partial_r v^k_r|^2 + |\partial_z v^k_r|^2 + \frac{1+k^2}{r^2}|v^k_r|^2 + \frac{2ik}{r^2} v^k_\theta \bar{v}^k_r \\ & + |\partial_r v^k_\theta|^2 + |\partial_z v^k_\theta|^2 + \frac{1+k^2}{r^2}|v^k_\theta|^2 - \frac{2ik}{r^2} v^k_r \bar{v}^k_\theta \\ & + |\partial_r v^k_z|^2 + |\partial_z v^k_z|^2 + \frac{k^2}{r^2}|v^k_z|^2 \Big) r\,\mathrm{d}r\,\mathrm{d}z. \end{aligned} \tag{5.16}$$

To see that (5.16) satisfies the properties of a norm, we consider the function $\underline{\breve{v}}^k \in (H^1(\breve{\Omega}))^3$ corresponding to a single Fourier coefficient $\underline{v}^k \in \boldsymbol{H}^1_{(k)}(\Omega)$:

$$\underline{\breve{v}}^k(x,y,z) := \frac{1}{\sqrt{2\pi}} \mathcal{R}_\theta \underline{v}^k(r,z) e^{ik\theta},$$

and, from (5.11), note that

$$\|\underline{v}^k\|_{\boldsymbol{H}^1_{(k)}(\Omega)} = \|\underline{\breve{v}}^k\|_{(H^1(\breve{\Omega}))^3}. \tag{5.17}$$

From (5.17) and the norm properties of $(H^1(\breve{\Omega}))^3$, we immediately get

$$\begin{aligned}
\|\underline{v}^k\|_{\boldsymbol{H}^1_{(k)}(\Omega)} &\geq 0,\\
\|\underline{v}^k\|_{\boldsymbol{H}^1_{(k)}(\Omega)} = 0 &\iff \|\underline{\breve{v}}^k\|_{(H^1(\breve{\Omega}))^3} = 0 \iff \underline{\breve{v}}^k = 0 \iff \underline{v}^k = 0,\\
\|c\underline{v}^k\|_{\boldsymbol{H}^1_{(k)}(\Omega)} &= \|c\underline{\breve{v}}^k\|_{(H^1(\breve{\Omega}))^3} = |c|\|\underline{\breve{v}}^k\|_{(H^1(\breve{\Omega}))^3} = |c|\|\underline{v}^k\|_{\boldsymbol{H}^1_{(k)}(\Omega)},\\
\|\underline{v}^k + \underline{w}^k\|_{\boldsymbol{H}^1_{(k)}(\Omega)} &= \|\underline{\breve{v}}^k + \underline{\breve{w}}^k\|_{(H^1(\breve{\Omega}))^3} \leq \|\underline{\breve{v}}^k\|_{(H^1(\breve{\Omega}))^3} + \|\underline{\breve{w}}^k\|_{(H^1(\breve{\Omega}))^3} = \|\underline{v}^k\|_{\boldsymbol{H}^1_{(k)}(\Omega)} + \|\underline{w}^k\|_{\boldsymbol{H}^1_{(k)}(\Omega)}.
\end{aligned}$$

Also from (5.17), the completeness of $\boldsymbol{H}^1_{(k)}(\Omega)$ is a consequence of the completeness of $(H^1(\breve{\Omega}))^3$.

We now characterize the spaces $\boldsymbol{H}^1_{(k)}(\Omega)$ in terms of the weighted spaces defined in Section 5.2. Beginning with the case $k = 0$, from (5.16),

$$\|\underline{v}^0\|^2_{\boldsymbol{H}^1_{(0)}(\Omega)} = \|\underline{v}^0\|^2_{(L^2_1(\Omega))^3} + \|v^0_r\|^2_{V^1_1(\Omega)} + \|v^0_\theta\|^2_{V^1_1(\Omega)} + |v^0_z|^2_{H^1_1(\Omega)}, \tag{5.18}$$

and we deduce that

$$\boldsymbol{H}^1_{(0)}(\Omega) = V^1_1(\Omega) \times V^1_1(\Omega) \times H^1_1(\Omega). \tag{5.19}$$

For the cases $k = \pm 1$, (5.16) gives

$$\begin{aligned}
\|\underline{v}^{\pm1}\|^2_{\boldsymbol{H}^1_{(\pm1)}(\Omega)} = \|\underline{v}^{\pm1}\|^2_{(L^2_1(\Omega))^3} &+ |v^{\pm1}_r|^2_{H^1_1(\Omega)} + |v^{\pm1}_\theta|^2_{H^1_1(\Omega)} + \|v^{\pm1}_z\|^2_{V^1_1(\Omega)}\\
&+ 2\int_\Omega \big(|v^{\pm1}_r|^2 + |v^{\pm1}_\theta|^2 \pm i(v^{\pm1}_\theta \bar{v}^{\pm1}_r - v^{\pm1}_r \bar{v}^{\pm1}_\theta)\big)\frac{1}{r}\,\mathrm{d}r\,\mathrm{d}z,
\end{aligned}$$

and by noting that

$$|v^{\pm1}_r \pm i v^{\pm1}_\theta|^2 = (v^{\pm1}_r \pm i v^{\pm1}_\theta)(\bar{v}^{\pm1}_r \mp i\bar{v}^{\pm1}_\theta) = |v^{\pm1}_r|^2 + |v^{\pm1}_\theta|^2 \pm i(v^{\pm1}_\theta \bar{v}^{\pm1}_r - v^{\pm1}_r \bar{v}^{\pm1}_\theta),$$

we obtain

$$\|\underline{v}^{\pm1}\|^2_{\boldsymbol{H}^1_{(\pm1)}(\Omega)} = \|\underline{v}^{\pm1}\|^2_{(L^2_1(\Omega))^3} + 2\|v^{\pm1}_r \pm i v^{\pm1}_\theta\|^2_{L^2_{-1}(\Omega)} + |v^{\pm1}_r|^2_{H^1_1(\Omega)} + |v^{\pm1}_\theta|^2_{H^1_1(\Omega)} + \|v^{\pm1}_z\|^2_{V^1_1(\Omega)}, \tag{5.20}$$

from which we deduce that, for $k = \pm 1$,

$$\boldsymbol{H}^1_{(k)}(\Omega) = \{\underline{v}^k \mid \underline{v}^k \in H^1_1(\Omega) \times H^1_1(\Omega) \times V^1_1(\Omega),\ v^k_r + ikv^k_\theta \in L^2_{-1}(\Omega)\}. \tag{5.21}$$

When $|k| \geq 2$, (5.16) gives

$$\begin{aligned}
\|\underline{v}^k\|^2_{\boldsymbol{H}^1_{(k)}(\Omega)} = \|\underline{v}^k\|^2_{(L^2_1(\Omega))^3} &+ |\underline{v}^k|^2_{(H^1_1(\Omega))^3} + (1 + k^2)(\|v^k_r\|^2_{L^2_{-1}(\Omega)} + \|v^k_\theta\|^2_{L^2_{-1}(\Omega)})\\
&+ k^2\|v^k_z\|^2_{L^2_{-1}(\Omega)} + 2ik\int_\Omega (v^k_\theta \bar{v}^k_r - v^k_r \bar{v}^k_\theta)\frac{1}{r}\,\mathrm{d}r\,\mathrm{d}z.
\end{aligned} \tag{5.22}$$

Noting that $i(v^k_\theta \bar{v}^k_r - v^k_r \bar{v}^k_\theta) = 2\,\mathrm{Im}(v^k_r \bar{v}^k_\theta)$, confirming that the right-hand sides are real for all $k \in \mathbb{Z}$, we can estimate the last term

$$\left|2ik\int_\Omega (v^k_\theta \bar{v}^k_r - v^k_r \bar{v}^k_\theta)\frac{1}{r}\,\mathrm{d}r\,\mathrm{d}z\right| \leq 4|k|\int_\Omega |v^k_r||v^k_\theta|\frac{1}{r}\,\mathrm{d}r\,\mathrm{d}z \leq 2|k|(\|v^k_r\|^2_{L^2_{-1}(\Omega)} + \|v^k_\theta\|^2_{L^2_{-1}(\Omega)}),$$

resulting in the inequalities

$$\begin{aligned}
&\|\underline{v}^k\|^2_{(L^2_1(\Omega))^3} + |\underline{v}^k|^2_{(H^1_1(\Omega))^3} + (|k| - 1)^2(\|v^k_r\|^2_{L^2_{-1}(\Omega)} + \|v^k_\theta\|^2_{L^2_{-1}(\Omega)}) + k^2\|v^k_z\|^2_{L^2_{-1}(\Omega)}\\
&\quad \leq \|\underline{v}^k\|^2_{\boldsymbol{H}^1_{(k)}(\Omega)} \leq \|\underline{v}^k\|^2_{(L^2_1(\Omega))^3} + |\underline{v}^k|^2_{(H^1_1(\Omega))^3} + (|k| + 1)^2(\|v^k_r\|^2_{L^2_{-1}(\Omega)} + \|v^k_\theta\|^2_{L^2_{-1}(\Omega)}) + k^2\|v^k_z\|^2_{L^2_{-1}(\Omega)}.
\end{aligned} \tag{5.23}$$

Since $|k| \geq 2$,

$$(|k| - 1)^2 = \left(1 - \frac{1}{|k|}\right)^2 k^2 \geq \frac{1}{4}k^2, \quad (|k| + 1)^2 = \left(1 + \frac{1}{|k|}\right)^2 k^2 \leq \frac{9}{4}k^2,$$

which, combined with (5.23), shows that the norm $\|\cdot\|_{\boldsymbol{H}^1_{(k)}(\Omega)}$ is equivalent to the norm $\|\cdot\|_{\boldsymbol{H}^1_{(k)*}(\Omega)}$:

$$\frac{1}{2}\|\underline{v}^k\|_{\boldsymbol{H}^1_{(k)*}(\Omega)} \le \|\underline{v}^k\|_{\boldsymbol{H}^1_{(k)}(\Omega)} \le \frac{3}{2}\|\underline{v}^k\|_{\boldsymbol{H}^1_{(k)*}(\Omega)},$$

where $\|\cdot\|_{\boldsymbol{H}^1_{(k)*}(\Omega)}$ is defined by

$$\|\underline{v}^k\|^2_{\boldsymbol{H}^1_{(k)*}(\Omega)} := \|\underline{v}^k\|^2_{(L^2_1(\Omega))^3} + |\underline{v}^k|^2_{(H^1_1(\Omega))^3} + k^2\|\underline{v}^k\|^2_{(L^2_{-1}(\Omega))^3}. \tag{5.24}$$

We deduce that, for $|k| \ge 2$,

$$\boldsymbol{H}^1_{(k)}(\Omega) = V^1_1(\Omega) \times V^1_1(\Omega) \times V^1_1(\Omega). \tag{5.25}$$

We summarize our results in the following characterization of $(H^1(\breve{\Omega}))^3$.

Theorem 5.4. *The mapping $\breve{\underline{v}} \mapsto (\underline{v}^k)_{k\in\mathbb{Z}}$, defined by* (5.3) *and* (5.6), *is an isometric isomorphism between $(H^1(\breve{\Omega}))^3$ and the l_2-sum $\bigoplus_2\{\boldsymbol{H}^1_{(k)}(\Omega) \mid k \in \mathbb{Z}\}$:*

$$\|\breve{\underline{v}}\|_{(H^1(\breve{\Omega}))^3} = \Big(\sum_{k\in\mathbb{Z}} \|\underline{v}^k\|^2_{\boldsymbol{H}^1_{(k)}(\Omega)}\Big)^{\frac{1}{2}} = \|(\underline{v}^k)\|_{\bigoplus_2\{\boldsymbol{H}^1_{(k)}(\Omega)|k\in\mathbb{Z}\}}, \tag{5.26}$$

where, from (5.19), (5.21), (5.25),

$$\boldsymbol{H}^1_{(k)}(\Omega) := \begin{cases} V^1_1(\Omega) \times V^1_1(\Omega) \times H^1_1(\Omega) & \text{if } k = 0,\\ \{\underline{v}^k \in H^1_1(\Omega) \times H^1_1(\Omega) \times V^1_1(\Omega),\ v^k_r + ikv^k_\theta \in L^2_{-1}(\Omega)\} & \text{if } |k| = 1,\\ V^1_1(\Omega) \times V^1_1(\Omega) \times V^1_1(\Omega) & \text{if } |k| \ge 2, \end{cases}$$

and where the norms $\|\cdot\|_{\boldsymbol{H}^1_{(k)}(\Omega)}$ are given by (5.18), (5.20), (5.22)*:*

$$\|\underline{v}^k\|^2_{\boldsymbol{H}^1_{(k)}(\Omega)} := \begin{cases} \|\underline{v}^k\|^2_{(L^2_1(\Omega))^3} + \|v^k_r\|^2_{V^1_1(\Omega)} + \|v^k_\theta\|^2_{V^1_1(\Omega)} + |v^k_z|^2_{H^1_1(\Omega)} & \text{if } k = 0,\\ \|\underline{v}^k\|^2_{(L^2_1(\Omega))^3} + 2\|v^k_r + ikv^k_\theta\|^2_{L^2_{-1}(\Omega)} + |v^k_r|^2_{H^1_1(\Omega)} + |v^k_\theta|^2_{H^1_1(\Omega)} + \|v^k_z\|^2_{V^1_1(\Omega)} & \text{if } |k| = 1,\\ \|\underline{v}^k\|^2_{(L^2_1(\Omega))^3} + |\underline{v}^k|^2_{(H^1_1(\Omega))^3} + (1+k^2)(\|v^k_r\|^2_{L^2_{-1}(\Omega)} + \|v^k_\theta\|^2_{L^2_{-1}(\Omega)}) & \\ \qquad + k^2\|v^k_z\|^2_{L^2_{-1}(\Omega)} + 2ik\int_\Omega (v^k_\theta\bar{v}^k_r - v^k_r\bar{v}^k_\theta)\frac{1}{r}\,\mathrm{d}r\,\mathrm{d}z & \text{if } |k| \ge 2. \end{cases}$$

For $|k| \ge 2$, we can use the equivalent norm $\|\cdot\|_{\boldsymbol{H}^1_{(k)*}(\Omega)}$ defined by (5.24)*:*

$$\|\underline{v}^k\|^2_{\boldsymbol{H}^1_{(k)*}(\Omega)} := \|\underline{v}^k\|^2_{(L^2_1(\Omega))^3} + |\underline{v}^k|^2_{(H^1_1(\Omega))^3} + k^2\|\underline{v}^k\|^2_{(L^2_{-1}(\Omega))^3}.$$

Remark 5.5. Corresponding to (5.26), we restate (5.9) in terms of the inner products $(\cdot,\cdot)_{\boldsymbol{H}^1_{(k)}(\Omega)}$ associated with the norms $\|\cdot\|_{\boldsymbol{H}^1_{(k)}(\Omega)}$:

$$(\breve{\underline{v}}, \breve{\underline{w}})_{(H^1(\breve{\Omega}))^3} = \sum_{k\in\mathbb{Z}} (\underline{v}^k, \underline{w}^k)_{\boldsymbol{H}^1_{(k)}(\Omega)}.$$

Remark 5.6. For comparison, in Appendix A, we derive (5.18), (5.20), (5.22) by an alternative method used in [5].

We now consider the subspace $(H^1_0(\breve{\Omega}))^3$ of $(H^1(\breve{\Omega}))^3$. The corresponding subspaces of $\boldsymbol{H}^1_{(k)}(\Omega)$ are

$$\boldsymbol{H}^1_{(k)\diamond}(\Omega) := \boldsymbol{H}^1_{(k)}(\Omega) \cap (H^1_{1\diamond}(\Omega))^3,$$

consisting of vector functions in $\boldsymbol{H}^1_{(k)}(\Omega)$ that vanish on the part $\Gamma = \partial\Omega\backslash\Gamma_0$ of the boundary that is not contained in the z-axis. Since we use the semi-norm $|\cdot|_{(H^1(\breve{\Omega}))^3}$ as a norm, equivalent to $\|\cdot\|_{(H^1(\breve{\Omega}))^3}$, on $(H^1_0(\breve{\Omega}))^3$, we state the following representation of $|\cdot|_{(H^1(\breve{\Omega}))^3}$, which follows immediately from (5.11) by omitting the terms corresponding to the $(L^2(\breve{\Omega}))^3$-norm:

$$|\breve{\underline{v}}|^2_{(H^1(\breve{\Omega}))^3} = \sum_{k\in\mathbb{Z}} |\underline{v}^k|^2_{\boldsymbol{H}^1_{(k)}(\Omega)}, \tag{5.27}$$

where we define

$$|\underline{v}^k|^2_{\boldsymbol{H}^1_{(k)}(\Omega)} := \int_\Omega \Big(|\partial_r v_r^k|^2 + |\partial_z v_r^k|^2 + \frac{1+k^2}{r^2}|v_r^k|^2 + \frac{2ik}{r^2} v_\theta^k \bar{v}_r^k \\ + |\partial_r v_\theta^k|^2 + |\partial_z v_\theta^k|^2 + \frac{1+k^2}{r^2}|v_\theta^k|^2 - \frac{2ik}{r^2} v_r^k \bar{v}_\theta^k \\ + |\partial_r v_z^k|^2 + |\partial_z v_z^k|^2 + \frac{k^2}{r^2}|v_z^k|^2 \Big) r\,\mathrm{d}r\,\mathrm{d}z.$$

Remark 5.7. The relation corresponding to (5.27) between the associated inner products $(\cdot,\cdot)_{(H^1_0(\breve{\Omega}))^3}$ and $(\cdot,\cdot)_{\boldsymbol{H}^1_{(k)\diamond}(\Omega)}$ is

$$(\breve{\underline{v}}, \breve{\underline{w}})_{(H^1_0(\breve{\Omega}))^3} = \sum_{k\in\mathbb{Z}} (\underline{v}^k, \underline{w}^k)_{\boldsymbol{H}^1_{(k)\diamond}(\Omega)}. \tag{5.28}$$

For all $k \in \mathbb{Z}$, $|\cdot|_{\boldsymbol{H}^1_{(k)}(\Omega)}$ is a norm, equivalent to $\|\cdot\|_{\boldsymbol{H}^1_{(k)}(\Omega)}$, on $\boldsymbol{H}^1_{(k)\diamond}(\Omega)$. This follows by again considering a function $\breve{\underline{v}}^k \in (H^1_0(\breve{\Omega}))^3$ corresponding to a single Fourier coefficient $\underline{v}^k \in \boldsymbol{H}^1_{(k)\diamond}(\Omega)$:

$$\breve{\underline{v}}^k(x,y,z) := \frac{1}{\sqrt{2\pi}} \mathcal{R}_\theta \underline{v}^k(r,z) e^{ik\theta},$$

and noting that, by (5.27), the equivalence between $|\cdot|_{(H^1(\breve{\Omega}))^3}$ and $\|\cdot\|_{(H^1(\breve{\Omega}))^3}$ on $(H^1_0(\breve{\Omega}))^3$, and (5.26),

$$|\underline{v}^k|_{\boldsymbol{H}^1_{(k)}(\Omega)} = |\breve{\underline{v}}^k|_{(H^1(\breve{\Omega}))^3} \geq c\|\breve{\underline{v}}^k\|_{(H^1(\breve{\Omega}))^3} = c\|\underline{v}^k\|_{\boldsymbol{H}^1_{(k)}(\Omega)}.$$

We get the following characterization of $(H^1_0(\breve{\Omega}))^3$.

Corollary 5.8. *The mapping $\breve{\underline{v}} \mapsto (\underline{v}^k)_{k\in\mathbb{Z}}$, defined by* (5.3) *and* (5.6), *is an isometric isomorphism between $(H^1_0(\breve{\Omega}))^3$ and the l_2-sum $\bigoplus_2\{\boldsymbol{H}^1_{(k)\diamond}(\Omega) \mid k \in \mathbb{Z}\}$:*

$$|\breve{\underline{v}}|_{(H^1(\breve{\Omega}))^3} = \Big(\sum_{k\in\mathbb{Z}} |\underline{v}^k|^2_{\boldsymbol{H}^1_{(k)}(\Omega)}\Big)^{\frac{1}{2}} = \|(\underline{v}^k)\|_{\bigoplus_2\{\boldsymbol{H}^1_{(k)\diamond}(\Omega)|k\in\mathbb{Z}\}}, \tag{5.29}$$

where

$$\boldsymbol{H}^1_{(k)\diamond}(\Omega) := \begin{cases} V^1_{1\diamond}(\Omega) \times V^1_{1\diamond}(\Omega) \times H^1_{1\diamond}(\Omega) & \text{if } k = 0, \\ \{\underline{v}^k \in H^1_{1\diamond}(\Omega) \times H^1_{1\diamond}(\Omega) \times V^1_{1\diamond}(\Omega),\ v_r^k + ikv_\theta^k \in L^2_{-1}(\Omega)\} & \text{if } |k| = 1, \\ V^1_{1\diamond}(\Omega) \times V^1_{1\diamond}(\Omega) \times V^1_{1\diamond}(\Omega) & \text{if } |k| \geq 2, \end{cases}$$

and

$$|\underline{v}^k|^2_{\boldsymbol{H}^1_{(k)}(\Omega)} := \begin{cases} \|v_r^k\|^2_{V^1_1(\Omega)} + \|v_\theta^k\|^2_{V^1_1(\Omega)} + |v_z^k|^2_{H^1_1(\Omega)} & \text{if } k = 0, \\ |v_r^k|^2_{H^1_1(\Omega)} + |v_\theta^k|^2_{H^1_1(\Omega)} + \|v_z^k\|^2_{V^1_1(\Omega)} + 2\|v_r^k + ikv_\theta^k\|^2_{L^2_{-1}(\Omega)} & \text{if } |k| = 1, \\ |\underline{v}^k|^2_{(H^1_1(\Omega))^3} + (1+k^2)(\|v_r^k\|^2_{L^2_{-1}(\Omega)} + \|v_\theta^k\|^2_{L^2_{-1}(\Omega)}) & \\ \qquad + k^2\|v_z^k\|^2_{L^2_{-1}(\Omega)} + 2ik\int_\Omega (v_\theta^k \bar{v}_r^k - v_r^k \bar{v}_\theta^k)\frac{1}{r}\,\mathrm{d}r\,\mathrm{d}z & \text{if } |k| \geq 2. \end{cases}$$

For $|k| \geq 2$, we can use the equivalent norm $|\cdot|_{\boldsymbol{H}^1_{(k)*}(\Omega)}$ defined by

$$|\underline{v}^k|^2_{\boldsymbol{H}^1_{(k)*}(\Omega)} := |\underline{v}^k|^2_{(H^1_1(\Omega))^3} + k^2\|\underline{v}^k\|^2_{(L^2_{-1}(\Omega))^3}.$$

5.5 Characterization of $(H^{-1}(\breve{\Omega}))^3$

For $\breve{\underline{f}} \in (H^{-1}(\breve{\Omega}))^3$, from the Riesz representation theorem, we deduce the existence of (a unique)

$$\breve{\underline{w}}_f \in (H^1_0(\breve{\Omega}))^3, \quad \text{with } \|\breve{\underline{f}}\|_{(H^{-1}(\breve{\Omega}))^3} = |\breve{\underline{w}}_f|_{(H^1(\breve{\Omega}))^3},$$

such that, for all $\breve{\underline{v}} \in (H^1_0(\breve{\Omega}))^3$,

$$\langle \breve{\underline{f}}, \breve{\underline{v}} \rangle_{(H^{-1}(\breve{\Omega}))^3 \times (H^1_0(\breve{\Omega}))^3} = (\breve{\underline{w}}_f, \breve{\underline{v}})_{(H^1_0(\breve{\Omega}))^3} = \sum_{k\in\mathbb{Z}} (\underline{w}_f^k, \underline{v}^k)_{\boldsymbol{H}^1_{(k)\diamond}(\Omega)} := \sum_{k\in\mathbb{Z}} \langle \underline{f}^k, \underline{v}^k \rangle_{\boldsymbol{H}^{-1}_{(k)}(\Omega) \times \boldsymbol{H}^1_{(k)\diamond}(\Omega)}, \tag{5.30}$$

where we have used (5.28), and $\boldsymbol{H}^{-1}_{(k)}(\Omega)$ denotes the dual space of $\boldsymbol{H}^1_{(k)\diamond}(\Omega)$.

Remark 5.9. For $\breve{\underline{f}} \in (L^2(\breve{\Omega}))^3$, we have

$$\underline{f}^k(r,z) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} (\mathcal{R}_{-\theta}\breve{\underline{f}})(r,\theta,z)e^{-ik\theta}\,\mathrm{d}\theta,$$

and we can write the duality pairings as integrals

$$\begin{aligned} \langle \breve{\underline{f}}, \breve{\underline{v}} \rangle_{(H^{-1}(\breve{\Omega}))^3 \times (H_0^1(\breve{\Omega}))^3} &= \int_{\breve{\Omega}} (\breve{\underline{f}} \cdot \bar{\breve{\underline{v}}})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \\ \langle \underline{f}^k, \underline{v}^k \rangle_{\boldsymbol{H}_{(k)}^{-1}(\Omega) \times \boldsymbol{H}_{(k)\diamond}^1(\Omega)} &= \int_{\Omega} (\underline{f}^k \cdot \bar{\underline{v}}^k) r\,\mathrm{d}r\,\mathrm{d}z. \end{aligned} \tag{5.31}$$

From (5.29) and the definition (5.30) of $\underline{f}^k$, we get

$$\|\breve{\underline{f}}\|^2_{(H^{-1}(\breve{\Omega}))^3} = |\breve{\underline{w}}_f|^2_{(H^1(\breve{\Omega}))^3} = \sum_{k\in\mathbb{Z}} |\underline{w}_f^k|^2_{\boldsymbol{H}^1_{(k)}(\Omega)} = \sum_{k\in\mathbb{Z}} \|\underline{f}^k\|^2_{\boldsymbol{H}^{-1}_{(k)}(\Omega)},$$

which gives the following characterization of $(H^{-1}(\breve{\Omega}))^3$.

Theorem 5.10. *The mapping* $\breve{\underline{f}} \mapsto (\underline{f}^k)_{k\in\mathbb{Z}}$*, defined by* (5.30)*, is an isometric isomorphism between* $(H^{-1}(\breve{\Omega}))^3$ *and the* l_2*-sum* $\bigoplus_2\{\boldsymbol{H}^{-1}_{(k)}(\Omega) \mid k \in \mathbb{Z}\}$*:*

$$\|\breve{\underline{f}}\|_{(H^{-1}(\breve{\Omega}))^3} = \Big(\sum_{k\in\mathbb{Z}} \|\underline{f}^k\|^2_{\boldsymbol{H}^{-1}_{(k)}(\Omega)}\Big)^{\frac{1}{2}} = \|(\underline{f}^k)\|_{\bigoplus_2\{\boldsymbol{H}^{-1}_{(k)}(\Omega)|k\in\mathbb{Z}\}}.$$

6 Two-Dimensional Problems

In this section, we state variational formulations of the two-dimensional problems for the Fourier coefficients, which are set in the spaces from the previous section. We show existence, uniqueness and stability from inf-sup conditions. We conclude by discussing some features of the special case with axisymmetric data.

6.1 Variational Formulation

Introducing, for all $k \in \mathbb{Z}$, the sesquilinear forms

$$\begin{aligned} \mathcal{A}_k(\underline{u}, \underline{v}) &:= (\underline{u}, \underline{v})_{\boldsymbol{H}^1_{(k)\diamond}(\Omega)} \\ &= \int_{\Omega} \Big(\partial_r u_r \partial_r \bar{v}_r + \partial_z u_r \partial_z \bar{v}_r + \frac{1+k^2}{r^2} u_r \bar{v}_r + \frac{2ik}{r^2} u_\theta \bar{v}_r \\ &\qquad + \partial_r u_\theta \partial_r \bar{v}_\theta + \partial_z u_\theta \partial_z \bar{v}_\theta + \frac{1+k^2}{r^2} u_\theta \bar{v}_\theta - \frac{2ik}{r^2} u_r \bar{v}_\theta \\ &\qquad + \partial_r u_z \partial_r \bar{v}_z + \partial_z u_z \partial_z \bar{v}_z + \frac{k^2}{r^2} u_z \bar{v}_z \Big) r\,\mathrm{d}r\,\mathrm{d}z, \\ \mathcal{B}_k(\underline{v}, q) &:= -\int_{\Omega} (\mathrm{div}_k\,\underline{v})\bar{q} r\,\mathrm{d}r\,\mathrm{d}z = -\int_{\Omega} (\partial_r(rv_r) + ikv_\theta + \partial_z(rv_z))\bar{q}\,\mathrm{d}r\,\mathrm{d}z, \end{aligned}$$

where div_k is defined by (4.6), we consider the following variational formulation of (4.5):

Find

$$(\underline{u}^k, p^k) \in \boldsymbol{H}^1_{(k)}(\Omega) \times L^2_{(k),0}(\Omega), \quad \text{where } \underline{u}^k - \underline{g}^k \in \boldsymbol{H}^1_{(k)\diamond}(\Omega),$$

such that, for all $(\underline{v}, q) \in \boldsymbol{H}^1_{(k)\diamond}(\Omega) \times L^2_{(k),0}(\Omega)$,

$$\begin{cases} \mathcal{A}_k(\underline{u}^k, \underline{v}) + \overline{\mathcal{B}_k(\underline{v}, p^k)} = \langle \underline{f}^k, \underline{v} \rangle_{\boldsymbol{H}^{-1}_{(k)}(\Omega) \times \boldsymbol{H}^1_{(k)\diamond}(\Omega)}, \\ \mathcal{B}_k(\underline{u}^k, q) = 0. \end{cases} \tag{6.1}$$

Remark 6.1. Note that the term $\mathcal{B}_k$ is conjugated in the first equation in (6.1), making the sesquilinear form

$$\mathcal{M}_k((\underline{u}, p), (\underline{v}, q)) := \mathcal{A}_k(\underline{u}, \underline{v}) + \overline{\mathcal{B}_k(\underline{v}, p)} + \mathcal{B}_k(\underline{u}, q)$$

Hermitian.

Remark 6.2. We use the same notation $\underline{g}^k$ for the Fourier coefficients of the Dirichlet boundary data $\underline{\breve{g}}$, and their liftings in $\boldsymbol{H}^1_{(k)}(\Omega)$. Existence of these liftings follows from the three-dimensional trace theorem since $\underline{\breve{g}} \in (H^{\frac{1}{2}}(\partial\breve{\Omega}))^3$ and thus, again using the same notation, admits a lifting $\underline{\breve{g}} \in (H^1(\breve{\Omega}))^3$.

Remark 6.3. For $\underline{\breve{f}} \in (L^2(\breve{\Omega}))^3$, we can, as noted in (5.31), write the duality pairings as integrals

$$\langle \underline{f}^k, \underline{v} \rangle_{\boldsymbol{H}^{-1}_{(k)}(\Omega) \times \boldsymbol{H}^1_{(k)\diamond}(\Omega)} = \int_\Omega (\underline{f}^k \cdot \underline{\bar{v}}) r \, \mathrm{d}r \, \mathrm{d}z.$$

Remark 6.4. The sesquilinear forms $\mathcal{A}_k(\cdot\,,\cdot)$ and $\mathcal{B}_k(\cdot\,,\cdot)$ are related to the corresponding sesquilinear forms

$$\breve{a}(\underline{\breve{u}}, \underline{\breve{v}}) = (\underline{\breve{u}}, \underline{\breve{v}})_{(H^1_0(\breve{\Omega}))^3} = \int_{\breve{\Omega}} \underline{\underline{\operatorname{grad}}}\, \underline{\breve{u}} : \underline{\underline{\operatorname{grad}}}\, \bar{\underline{\breve{v}}} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z, \quad \breve{b}(\underline{\breve{v}}, \breve{q}) = -\int_{\breve{\Omega}} (\operatorname{div} \underline{\breve{v}}) \bar{\breve{q}} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z,$$

in the following mixed formulation of the three-dimensional problem (2.1):

Find $(\underline{\breve{u}}, \breve{p}) \in (H^1(\breve{\Omega}))^3 \times L^2_0(\breve{\Omega})$, where $\underline{\breve{u}} - \underline{\breve{g}} \in (H^1_0(\breve{\Omega}))^3$, such that, for all $(\underline{\breve{v}}, \breve{q}) \in (H^1_0(\breve{\Omega}))^3 \times L^2_0(\breve{\Omega})$,

$$\begin{cases} \breve{a}(\underline{\breve{u}}, \underline{\breve{v}}) + \overline{\breve{b}(\underline{\breve{v}}, \breve{p})} = \langle \underline{\breve{f}}, \underline{\breve{v}} \rangle_{(H^{-1}(\breve{\Omega}))^3 \times (H^1_0(\breve{\Omega}))^3}, \\ \breve{b}(\underline{\breve{u}}, \breve{q}) = 0, \end{cases}$$

by the relations

$$\breve{a}(\underline{\breve{u}}, \underline{\breve{v}}) = \sum_{k \in \mathbb{Z}} \mathcal{A}_k(\underline{u}^k, \underline{v}^k), \tag{6.2}$$

$$\breve{b}(\underline{\breve{v}}, \breve{q}) = \sum_{k \in \mathbb{Z}} \mathcal{B}_k(\underline{v}^k, q^k), \tag{6.3}$$

where (6.2) is a restatement of (5.28), and (6.3) is an analogous consequence of (3.6) and the orthogonality on $[-\pi, \pi]$ of the family $\{e^{ik\theta}\}_{k=-\infty}^{+\infty}$ of basis functions (cf. (5.7)).

6.2 Well-Posedness

Setting $\underline{u}^k = \underline{u}^k_0 + \underline{g}^k$, we rewrite (6.1) as follows:

Find $(\underline{u}^k_0, p^k) \in \boldsymbol{H}^1_{(k)\diamond}(\Omega) \times L^2_{(k),0}(\Omega)$ such that, for all $(\underline{v}, q) \in \boldsymbol{H}^1_{(k)\diamond}(\Omega) \times L^2_{(k),0}(\Omega)$,

$$\begin{cases} \mathcal{A}_k(\underline{u}^k_0, \underline{v}) + \overline{\mathcal{B}_k(\underline{v}, p^k)} = \langle \underline{f}^k, \underline{v} \rangle_{\boldsymbol{H}^{-1}_{(k)}(\Omega) \times \boldsymbol{H}^1_{(k)\diamond}(\Omega)} - \mathcal{A}_k(\underline{g}^k, \underline{v}), \\ \mathcal{B}_k(\underline{u}^k_0, q) = -\mathcal{B}_k(\underline{g}^k, q). \end{cases}$$

Since $\mathcal{A}_k(\cdot\,,\cdot)$ is a nonnegative, Hermitian form, where $\mathcal{A}_k(\underline{v}, \underline{v}) = |\underline{v}|^2_{\boldsymbol{H}^1_{(k)}(\Omega)}$, we have

$$|\mathcal{A}_k(\underline{u}, \underline{v})| \le \mathcal{A}_k(\underline{u}, \underline{u})^{\frac{1}{2}} \mathcal{A}_k(\underline{v}, \underline{v})^{\frac{1}{2}} = |\underline{u}|_{\boldsymbol{H}^1_{(k)}(\Omega)} |\underline{v}|_{\boldsymbol{H}^1_{(k)}(\Omega)},$$

showing coercivity of $\mathcal{A}_k(\cdot\,,\cdot)$ on $\boldsymbol{H}^1_{(k)\diamond}(\Omega)$, continuity of $\mathcal{A}_k(\cdot\,,\cdot)$ on $\boldsymbol{H}^1_{(k)\diamond}(\Omega) \times \boldsymbol{H}^1_{(k)\diamond}(\Omega)$, and

$$\mathcal{A}_k(\underline{g}^k, \cdot) \in \boldsymbol{H}^{-1}_{(k)}(\Omega) \quad \text{with } \|\mathcal{A}_k(\underline{g}^k, \cdot)\|_{\boldsymbol{H}^{-1}_{(k)}(\Omega)} \le |\underline{g}^k|_{\boldsymbol{H}^1_{(k)}(\Omega)}.$$

Again considering functions corresponding to a single Fourier coefficient

$$\underline{\breve{v}}^k(x, y, z) = \frac{1}{\sqrt{2\pi}} \mathcal{R}_\theta \underline{v}^k(r, z) e^{ik\theta},$$

$$\breve{q}^k(x, y, z) = \frac{1}{\sqrt{2\pi}} q^k(r, z) e^{ik\theta}, \tag{6.4}$$

we note from (6.3), the continuity of the three-dimensional form $\breve{b}(\cdot,\cdot)$, (5.27) and (5.15) that

$$|\mathcal{B}_k(\underline{v}^k, q^k)| = |\breve{b}(\breve{\underline{v}}^k, \breve{q}^k)| \le \sqrt{3}|\breve{\underline{v}}^k|_{(H^1(\breve{\Omega}))^3}\|\breve{q}^k\|_{L^2(\breve{\Omega})} = \sqrt{3}|\underline{v}^k|_{\boldsymbol{H}^1_{(k)}(\Omega)}\|q^k\|_{L^2_1(\Omega)},$$

which shows continuity of $\mathcal{B}_k(\cdot,\cdot)$ on $\boldsymbol{H}^1_{(k)\diamond}(\Omega) \times L^2_{(k),0}(\Omega)$, and

$$\mathcal{B}_k(\underline{g}^k, \cdot) \in L^2_{(k),0}(\Omega)^* \quad \text{with } \|\mathcal{B}_k(\underline{g}^k,\cdot)\|_{L^2_{(k),0}(\Omega)^*} \le \sqrt{3}|\underline{g}^k|_{\boldsymbol{H}^1_{(k)}(\Omega)}.$$

The inf-sup condition for $\mathcal{B}_k(\cdot,\cdot)$ on $\boldsymbol{H}^1_{(k)\diamond}(\Omega) \times L^2_{(k),0}(\Omega)$ that there exists a positive constant β (independent of k) such that,

$$\text{for all } q^k \in L^2_{(k),0}(\Omega), \qquad \sup_{\underline{v}^k \in \boldsymbol{H}^1_{(k)\diamond}(\Omega)} \frac{|\mathcal{B}_k(\underline{v}^k, q^k)|}{|\underline{v}^k|_{\boldsymbol{H}^1_{(k)}(\Omega)}} \ge \beta\|q^k\|_{L^2_1(\Omega)}$$

follows from the three-dimensional inf-sup condition, from which, for an arbitrary $q^k \in L^2_{(k),0}(\Omega)$ and corresponding $\breve{q}^k \in L^2_0(\breve{\Omega})$ given by (6.4), we deduce that there exists $\breve{\underline{w}} \in (H^1_0(\breve{\Omega}))^3$ such that, for its k-th Fourier coefficient $\underline{w}^k \in \boldsymbol{H}^1_{(k)\diamond}(\Omega)$,

$$\frac{|\mathcal{B}_k(\underline{w}^k, q^k)|}{|\underline{w}^k|_{\boldsymbol{H}^1_{(k)}(\Omega)}} \ge \frac{|\mathcal{B}_k(\underline{w}^k, q^k)|}{|\breve{\underline{w}}|_{(H^1(\breve{\Omega}))^3}} = \frac{|\breve{b}(\breve{\underline{w}}, \breve{q}^k)|}{|\breve{\underline{w}}|_{(H^1(\breve{\Omega}))^3}} \ge \beta\|\breve{q}^k\|_{L^2(\breve{\Omega})} = \beta\|q^k\|_{L^2_1(\Omega)}.$$

We have thus proved (see, e.g., [4, Theorem 4.2.3]) the following theorem.

Theorem 6.5. *For $\breve{\underline{f}} \in (H^{-1}(\breve{\Omega}))^3$ and $\breve{\underline{g}} \in (H^{\frac{1}{2}}(\partial\breve{\Omega}))^3$, where $\underline{g}^0$ satisfies the compatibility condition* (4.7)*, the variational formulations* (6.1) *have unique solutions for all $k \in \mathbb{Z}$. The solutions are bounded, uniformly in k, by*

$$\|\underline{u}^k\|_{\boldsymbol{H}^1_{(k)}(\Omega)} + \|p^k\|_{L^2_1(\Omega)} \le C(\|\underline{f}^k\|_{\boldsymbol{H}^{-1}_{(k)}(\Omega)} + \|\underline{g}^k\|_{\boldsymbol{H}^1_{(k)}(\Omega)}). \tag{6.5}$$

6.3 Axisymmetric Data

For axisymmetric data, i.e., only $\underline{f}^0$ and $\underline{g}^0$ are non-zero, it follows immediately from uniqueness that only $\underline{u}^0$ and p^0 are non-zero, which means that the solution will also be axisymmetric. Also note that problem (6.1) for $k = 0$ decouples into two independent problems: one for (u^0_r, u^0_z, p^0) and one for u^0_θ. If the data are real-valued, these problems have real-valued solutions.

7 Anisotropic Spaces

We introduce two families $\boldsymbol{H}^{\pm1,s}(\breve{\Omega})$ of anisotropic spaces, where s is a nonnegative real number measuring "extra regularity" in the angular direction, that we (following [3]) will use to derive an estimate of the error due to Fourier truncation in Section 8.

When s is a nonnegative integer, we define

$$\boldsymbol{H}^{\pm1,s}(\breve{\Omega}) := \{\breve{\underline{v}} \mid \mathcal{R}_\theta\partial^l_\theta\mathcal{R}_{-\theta}\breve{\underline{v}} \in (H^{\pm1}(\breve{\Omega}))^3,\ 0 \le l \le s\},$$

equipped with the norms

$$\|\breve{\underline{v}}\|_{\boldsymbol{H}^{\pm1,s}(\breve{\Omega})} := \Big(\sum_{k\in\mathbb{Z}}(1+k^2)^s\|\underline{v}^k\|^2_{\boldsymbol{H}^{\pm1}_{(k)}(\Omega)}\Big)^{\frac{1}{2}}. \tag{7.1}$$

Note that $\boldsymbol{H}^{\pm1,0}(\breve{\Omega}) = (H^{\pm1}(\breve{\Omega}))^3$. The norms defined by (7.1) are equivalent to the natural norms

$$\Big(\sum_{l=0}^{s}\|\mathcal{R}_\theta\partial^l_\theta\mathcal{R}_{-\theta}\breve{\underline{v}}\|^2_{(H^{\pm1}(\breve{\Omega}))^3}\Big)^{\frac{1}{2}}, \tag{7.2}$$

as we show in the following lemma.

Lemma 7.1. *For any nonnegative integer s, the norms defined by* (7.1) *are equivalent to the natural norms defined by* (7.2).

Proof. Let $\underline{\breve{v}} \in \boldsymbol{H}^{\pm 1,s}(\breve{\Omega})$ and $0 \le l \le s$. From the Fourier expansions

$$\mathcal{R}_\theta \partial_\theta^l \mathcal{R}_{-\theta} \underline{\breve{v}}(x, y, z) = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} \mathcal{R}_\theta (ik)^l \underline{v}^k(r, z) e^{ik\theta}$$

and Theorem 5.4/5.10 (for $\boldsymbol{H}^{1,s}(\breve{\Omega})$ and $\boldsymbol{H}^{-1,s}(\breve{\Omega})$, respectively), we get

$$\sum_{l=0}^{s} \|\mathcal{R}_\theta \partial_\theta^l \mathcal{R}_{-\theta} \underline{\breve{v}}\|^2_{(H^{\pm 1}(\breve{\Omega}))^3} = \sum_{l=0}^{s} \sum_{k \in \mathbb{Z}} \|(ik)^l \underline{v}^k\|^2_{\boldsymbol{H}^{\pm 1}_{(k)}(\Omega)} = \sum_{k \in \mathbb{Z}} \Big(\sum_{l=0}^{s} k^{2l} \Big) \|\underline{v}^k\|^2_{\boldsymbol{H}^{\pm 1}_{(k)}(\Omega)},$$

from which the result follows. □

We generalize by extending the norms defined by (7.1) to an arbitrary nonnegative real number s:

$$\boldsymbol{H}^{\pm 1,s}(\breve{\Omega}) := \Big\{ \underline{\breve{v}} \;\Big|\; \|\underline{\breve{v}}\|_{\boldsymbol{H}^{\pm 1,s}(\breve{\Omega})} := \Big(\sum_{k \in \mathbb{Z}} (1 + k^2)^s \|\underline{v}^k\|^2_{\boldsymbol{H}^{\pm 1}_{(k)}(\Omega)} \Big)^{\frac{1}{2}} < +\infty \Big\}. \tag{7.3}$$

8 Error Due to Fourier Truncation

Introducing the truncated Fourier series

$$\underline{\breve{u}}_{[N]} = \frac{1}{\sqrt{2\pi}} \sum_{|k| \le N} \mathcal{R}_\theta \underline{u}^k(r, z) e^{ik\theta}, \quad \breve{p}_{[N]} = \frac{1}{\sqrt{2\pi}} \sum_{|k| \le N} p^k(r, z) e^{ik\theta},$$

we prove the following estimate of the error due to Fourier truncation.

Theorem 8.1. *Let s be a nonnegative real number. If the data* $(\underline{\breve{f}}, \underline{\breve{g}})$ *of* (2.1) *belong to* $\boldsymbol{H}^{-1,s}(\breve{\Omega}) \times \boldsymbol{H}^{1,s}(\breve{\Omega})$, *the following error estimate holds between the solution* $(\underline{\breve{u}}, \breve{p})$ *and its truncated Fourier series* $(\underline{\breve{u}}_{[N]}, \breve{p}_{[N]})$*:*

$$\|\underline{\breve{u}} - \underline{\breve{u}}_{[N]}\|_{(H^1(\breve{\Omega}))^3} + \|\breve{p} - \breve{p}_{[N]}\|_{L^2(\breve{\Omega})} \le C N^{-s} (\|\underline{\breve{f}}\|_{\boldsymbol{H}^{-1,s}(\breve{\Omega})} + \|\underline{\breve{g}}\|_{\boldsymbol{H}^{1,s}(\breve{\Omega})}).$$

Proof. Using the isometries (5.15) and (5.26), together with the regularity estimate (6.5) and the definitions of the anisotropic norms (7.3), we have

$$\begin{aligned}
\|\underline{\breve{u}} - \underline{\breve{u}}_{[N]}\|^2_{(H^1(\breve{\Omega}))^3} + \|\breve{p} - \breve{p}_{[N]}\|^2_{L^2(\breve{\Omega})} &= \Big\| \frac{1}{\sqrt{2\pi}} \sum_{|k|>N} \mathcal{R}_\theta \underline{u}^k e^{ik\theta} \Big\|^2_{(H^1(\breve{\Omega}))^3} + \Big\| \frac{1}{\sqrt{2\pi}} \sum_{|k|>N} p^k e^{ik\theta} \Big\|^2_{L^2(\breve{\Omega})} \\
&= \sum_{|k|>N} (\|\underline{u}^k\|^2_{\boldsymbol{H}^1_{(k)}(\Omega)} + \|p^k\|^2_{L^2_1(\Omega)}) \\
&\le 2C^2 \sum_{|k|>N} (\|\underline{f}^k\|^2_{\boldsymbol{H}^{-1}_{(k)}(\Omega)} + \|\underline{g}^k\|^2_{\boldsymbol{H}^1_{(k)}(\Omega)}) \\
&\le 2C^2 N^{-2s} \sum_{|k|>N} (1 + k^2)^s (\|\underline{f}^k\|^2_{\boldsymbol{H}^{-1}_{(k)}(\Omega)} + \|\underline{g}^k\|^2_{\boldsymbol{H}^1_{(k)}(\Omega)}) \\
&\le 2C^2 N^{-2s} (\|\underline{\breve{f}}\|^2_{\boldsymbol{H}^{-1,s}(\breve{\Omega})} + \|\underline{\breve{g}}\|^2_{\boldsymbol{H}^{1,s}(\breve{\Omega})}).
\end{aligned}$$
□

Remark 8.2. As emphasized in [2], the error from Fourier truncation only depends on the regularity of the data $(\underline{\breve{f}}, \underline{\breve{g}})$, not on the regularity of the solution $(\underline{\breve{u}}, \breve{p})$ (which is geometry dependent). This means that, for regular (with respect to θ) data, a small value for N suffices.

A Comparison with Costabel, Dauge and Hu

In [5], characterization of $H^m(\breve{\Omega})$ by Fourier coefficients (for any positive integer m) is treated first; then the relation (see [5, Proposition 6.1])

$$\|\underline{v}^k\|^2_{\boldsymbol{H}^1_{(k)}(\Omega)} = \frac{1}{2}\|v^k_r + iv^k_\theta\|^2_{H^1_{(k+1)}(\Omega)} + \frac{1}{2}\|v^k_r - iv^k_\theta\|^2_{H^1_{(k-1)}(\Omega)} + \|v^k_z\|^2_{H^1_{(k)}(\Omega)}, \tag{A.1}$$

linking vector $\boldsymbol{H}^1_{(k)}(\Omega)$-norms and scalar $H^1_{(k)}(\Omega)$-norms, is used to derive characterizations of $(H^m(\breve{\Omega}))^3$.

In the present work, we have derived the characterization of $(H^1(\breve{\Omega}))^3$ in Theorem 5.4 by directly rewriting the $(H^1(\breve{\Omega}))^3$-norm, where the decomposition (5.11) resulted in the $\boldsymbol{H}^1_{(k)}(\Omega)$-norms given by (5.18) for $k = 0$, by (5.20) for $|k| = 1$, and by (5.22) for $|k| \geq 2$.

To compare the methods, we first note that

$$\|v\|^2_{H^1_{(k)}(\Omega)} := \begin{cases} \|v\|^2_{L^2_1(\Omega)} + |v|^2_{H^1_1(\Omega)} & \text{if } k = 0, \\ \|v\|^2_{L^2_1(\Omega)} + |v|^2_{H^1_1(\Omega)} + k^2\|v\|^2_{L^2_{-1}(\Omega)} & \text{if } |k| \geq 1, \end{cases} \tag{A.2}$$

which follows, as in Section 5.1, by expressing the $H^1(\breve{\Omega})$-inner product in cylindrical coordinates and using Fourier expansions:

$$\begin{aligned}
(\breve{u}, \breve{v})_{H^1(\breve{\Omega})} &= \int\limits_{\breve{\Omega}} (\breve{u}\bar{\breve{v}} + \partial_x\breve{u}\partial_x\bar{\breve{v}} + \partial_y\breve{u}\partial_y\bar{\breve{v}} + \partial_z\breve{u}\partial_z\bar{\breve{v}})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \int\limits_{\breve{\Omega}} \Big(u\bar{v} + \partial_r u\partial_r\bar{v} + \frac{1}{r^2}\partial_\theta u\partial_\theta\bar{v} + \partial_z u\partial_z\bar{v}\Big) r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z \\
&= \sum_{k\in\mathbb{Z}} \int\limits_{\Omega} \Big(u^k\bar{v}^k + \partial_r u^k\partial_r\bar{v}^k + \partial_z u^k\partial_z\bar{v}^k + \frac{k^2}{r^2}u^k\bar{v}^k\Big) r\,\mathrm{d}r\,\mathrm{d}z =: \sum_{k\in\mathbb{Z}} (u^k, v^k)_{H^1_{(k)}(\Omega)}.
\end{aligned}$$

From (A.1), (A.2), and the polarization identity

$$\|a + b\|^2 + \|a - b\|^2 = 2(\|a\|^2 + \|b\|^2),$$

we then get, for $k = 0$,

$$\begin{aligned}
\|\underline{v}^0\|^2_{\boldsymbol{H}^1_{(0)}(\Omega)} &= \frac{1}{2}\|v^0_r + iv^0_\theta\|^2_{H^1_{(1)}(\Omega)} + \frac{1}{2}\|v^0_r - iv^0_\theta\|^2_{H^1_{(-1)}(\Omega)} + \|v^0_z\|^2_{H^1_{(0)}(\Omega)} \\
&= \frac{1}{2}\big(\|v^0_r + iv^0_\theta\|^2_{L^2_1(\Omega)} + |v^0_r + iv^0_\theta|^2_{H^1_1(\Omega)} + \|v^0_r + iv^0_\theta\|^2_{L^2_{-1}(\Omega)} \\
&\qquad + \|v^0_r - iv^0_\theta\|^2_{L^2_1(\Omega)} + |v^0_r - iv^0_\theta|^2_{H^1_1(\Omega)} + \|v^0_r - iv^0_\theta\|^2_{L^2_{-1}(\Omega)}\big) + \|v^0_z\|^2_{L^2_1(\Omega)} + |v^0_z|^2_{H^1_1(\Omega)} \\
&= \|v^0_r\|^2_{L^2_1(\Omega)} + \|v^0_\theta\|^2_{L^2_1(\Omega)} + \|v^0_z\|^2_{L^2_1(\Omega)} + |v^0_r|^2_{H^1_1(\Omega)} + |v^0_\theta|^2_{H^1_1(\Omega)} + \|v^0_r\|^2_{L^2_{-1}(\Omega)} + \|v^0_\theta\|^2_{L^2_{-1}(\Omega)} + |v^0_z|^2_{H^1_1(\Omega)} \\
&= \|\underline{v}^0\|^2_{(L^2_1(\Omega))^3} + \|v^0_r\|^2_{V^1_1(\Omega)} + \|v^0_\theta\|^2_{V^1_1(\Omega)} + |v^0_z|^2_{H^1_1(\Omega)},
\end{aligned}$$

which is the same as (5.18).

Similarly, for $|k| = 1$, we get

$$\begin{aligned}
\|\underline{v}^{\pm1}\|^2_{\boldsymbol{H}^1_{(\pm1)}(\Omega)} &= \frac{1}{2}\|v^{\pm1}_r \pm iv^{\pm1}_\theta\|^2_{H^1_{(\pm2)}(\Omega)} + \frac{1}{2}\|v^{\pm1}_r \mp iv^{\pm1}_\theta\|^2_{H^1_{(0)}(\Omega)} + \|v^{\pm1}_z\|^2_{H^1_{(\pm1)}(\Omega)} \\
&= \frac{1}{2}\big(\|v^{\pm1}_r \pm iv^{\pm1}_\theta\|^2_{L^2_1(\Omega)} + \|v^{\pm1}_r \mp iv^{\pm1}_\theta\|^2_{L^2_1(\Omega)} + |v^{\pm1}_r \pm iv^{\pm1}_\theta|^2_{H^1_1(\Omega)} + |v^{\pm1}_r \mp iv^{\pm1}_\theta|^2_{H^1_1(\Omega)}\big) \\
&\qquad + 2\|v^{\pm1}_r \pm iv^{\pm1}_\theta\|^2_{L^2_{-1}(\Omega)} + \|v^{\pm1}_z\|^2_{L^2_1(\Omega)} + |v^{\pm1}_z|^2_{H^1_1(\Omega)} + \|v^{\pm1}_z\|^2_{L^2_{-1}(\Omega)} \\
&= \|v^{\pm1}_r\|^2_{L^2_1(\Omega)} + \|v^{\pm1}_\theta\|^2_{L^2_1(\Omega)} + |v^{\pm1}_r|^2_{H^1_1(\Omega)} + |v^{\pm1}_\theta|^2_{H^1_1(\Omega)} \\
&\qquad + 2\|v^{\pm1}_r \pm iv^{\pm1}_\theta\|^2_{L^2_{-1}(\Omega)} + \|v^{\pm1}_z\|^2_{L^2_1(\Omega)} + \|v^{\pm1}_z\|^2_{V^1_1(\Omega)} \\
&= \|\underline{v}^{\pm1}\|^2_{(L^2_1(\Omega))^3} + 2\|v^{\pm1}_r \pm iv^{\pm1}_\theta\|^2_{L^2_{-1}(\Omega)} + |v^{\pm1}_r|^2_{H^1_1(\Omega)} + |v^{\pm1}_\theta|^2_{H^1_1(\Omega)} + \|v^{\pm1}_z\|^2_{V^1_1(\Omega)},
\end{aligned}$$

which is the same as (5.20).

Finally, for $|k| \geq 2$ (noting that $|k \pm 1| \geq 1$), we get

$$\begin{aligned}
\|\underline{v}^k\|^2_{\boldsymbol{H}^1_{(k)}(\Omega)} &= \frac{1}{2}\|v_r^k + iv_\theta^k\|^2_{H^1_{(k+1)}(\Omega)} + \frac{1}{2}\|v_r^k - iv_\theta^k\|^2_{H^1_{(k-1)}(\Omega)} + \|v_z^k\|^2_{H^1_{(k)}(\Omega)} \\
&= \frac{1}{2}\big(\|v_r^k + iv_\theta^k\|^2_{L^2_1(\Omega)} + \|v_r^k - iv_\theta^k\|^2_{L^2_1(\Omega)} + |v_r^k + iv_\theta^k|^2_{H^1_1(\Omega)} + |v_r^k - iv_\theta^k|^2_{H^1_1(\Omega)} \\
&\qquad + (k+1)^2\|v_r^k + iv_\theta^k\|^2_{L^2_{-1}(\Omega)} + (k-1)^2\|v_r^k - iv_\theta^k\|^2_{L^2_{-1}(\Omega)}\big) + \|v_z^k\|^2_{L^2_1(\Omega)} + |v_z^k|^2_{H^1_1(\Omega)} + k^2\|v_z^k\|^2_{L^2_{-1}(\Omega)} \\
&= \|v_r^k\|^2_{L^2_1(\Omega)} + \|v_\theta^k\|^2_{L^2_1(\Omega)} + \|v_z^k\|^2_{L^2_1(\Omega)} + |v_r^k|^2_{H^1_1(\Omega)} + |v_\theta^k|^2_{H^1_1(\Omega)} + |v_z^k|^2_{H^1_1(\Omega)} \\
&\qquad + (1+k^2)\big(\|v_r^k\|^2_{L^2_{-1}(\Omega)} + \|v_\theta^k\|^2_{L^2_{-1}(\Omega)}\big) + k^2\|v_z^k\|^2_{L^2_{-1}(\Omega)} \\
&\qquad + k\big(\|v_r^k + iv_\theta^k\|^2_{L^2_{-1}(\Omega)} - \|v_r^k - iv_\theta^k\|^2_{L^2_{-1}(\Omega)}\big) \\
&= \|\underline{v}^k\|^2_{(L^2_1(\Omega))^3} + |\underline{v}^k|^2_{(H^1_1(\Omega))^3} + (1+k^2)\big(\|v_r^k\|^2_{L^2_{-1}(\Omega)} + \|v_\theta^k\|^2_{L^2_{-1}(\Omega)}\big) \\
&\qquad + k^2\|v_z^k\|^2_{L^2_{-1}(\Omega)} + 2ik\int_\Omega (v_\theta^k \bar{v}_r^k - v_r^k \bar{v}_\theta^k)\frac{1}{r}\,\mathrm{d}r\,\mathrm{d}z,
\end{aligned}$$

which is the same as (5.22).

In conclusion, the $\boldsymbol{H}^1_{(k)}(\Omega)$-norms for the Fourier coefficient spaces that we have derived by directly rewriting the $(H^1(\breve{\Omega}))^3$-norm are the same as those obtained from relation (A.1). In addition to conveying structural understanding, relation (A.1) also facilitates the general study, by induction, for positive integer order Sobolev spaces carried out in [5]. For our exposition, directly aimed at the Stokes problem, the method we have chosen is straightforward to follow. The same remark can be made as to the difference between introducing the differential operators $\partial_\zeta = \frac{1}{\sqrt{2}}(\partial_x - i\partial_y)$ and $\partial_{\bar{\zeta}} = \frac{1}{\sqrt{2}}(\partial_x + i\partial_y)$ in [5], and our choice of working directly with ∂_x and ∂_y.

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